Vibrations in Physical Systems Vol.26 (2014)

Nonlinear Vibrations of Periodic Beams

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Abstract

Geometrically nonlinear vibrations of beams with properties periodically varying along the axis are investigated. The tolerance method of averaging differential operators with highly oscillating coefficients is applied to obtain the governing equations with constant coefficients. The proposed model describes the dynamics of the beam with the effect of the microstructure size.

Keywords: nonlinear vibrations, periodic beams, tolerance modelling

1. Introduction

The note concerns with geometrically nonlinear vibrations of beams with periodically varying mass, geometric and material properties along the beam axis. Moreover, this beam can interact with periodically nonhomogeneous viscoelastic subsoil. A fragment of such beam is shown in Fig. 1. Equations of motion of such structures have usually non-continuous, highly oscillating, periodic coefficients. Since, various averaging methods which lead to approximate models, determined by equations with constant coefficients, are applied. Among them methods based on the asymptotic homogenization can be mentioned, cf. [3].

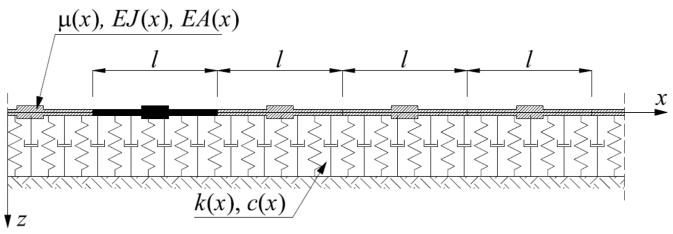

Figure 1. A fragment of a periodic beam

In this contribution, in order to replace the differential equations with highly oscillating coefficients by equations with constant coefficients, the tolerance modelling is applied. This approach was introduced for the purpose of analysis of various thermomechanical problems of periodic elastic composites, e.g. it was used to analyse vibrations of beams within the linear theory, cf. [5], where equations and their generalization by including influence of the axial force, an elastic subsoil and viscous damping have been derived in this way.

The main aim of this note is to derive the tolerance model equations with constant coefficients, which describe geometrically nonlinear vibrations of periodic beams resting on a periodic viscoelastic foundation, with taking into account the effect of the microstructure size.

2. Formulation of the problem

The object under consideration is a linearly elastic prismatic beam, bilateraly interacting with a periodic viscoelastic foundation. Let *Oxyz* be an orthogonal Cartesian coordinate system, the *Ox* axis coincides with the axis of the beam, the cross section of the beam be symmetric with respect to the plane of the load *Oxz*, the load acts in the direction of the axis *Oz*. The problem can be treated as one-dimensional.

The beam is assumed to be made of many repetitive small elements, called *periodicity cells*, each of which is defined as $\Delta\equiv[-l/2,l/2]$, where $l<<L$ is the length of the cell and named *the microstructure parameter*.

Our considerations are based on the Euler-Bernoulli theory of beams. Additionally large transverse deflection but small deformations are assumed, cf. [4]. The effects of axial and rotational inertia are neglected in further considerations. Let $\partial^k=\partial^k/\partial x^k$ be the k-th derivative of a function with respect to the x coordinate. Let the transverse deflection, the longitudinal displacement, tensile and flexural stiffness, the elastic coefficient of the foundation, the damping coefficient of the foundation, density of beam material per unit length, transverse load and dissipative force by $w = w(x,t)$, $u_0 = u_0(x,t)$, $EA = EA(x)$, $EJ = EJ(x)$, $k = k(x)$, $c = c(x)$, $\mu = \mu(x)$, $q = q(x,t)$, $p = p(x,t)$, the system of nonlinear coupled differential equations for the longitudinal displacements u_0 and the transverse deflection w can be written as:

$$\begin{aligned}&\partial[EA(\partial u_0+\tfrac{1}{2}(\partial w)^2)]=0,\\&\partial^2(EJ\partial^2 w)-EA[\partial u_0+\tfrac{1}{2}(\partial w)^2]\partial^2 w+c\dot{w}+\mu\ddot{w}=q.\end{aligned}\tag{1}$$

The coefficients EA, EJ, k, μ, c, and in some cases the load q, are highly oscillating, often non-continuous functions of the x coordinate.

3. Introductory concepts and basic assumptions of the tolerance modelling

The averaged equations of periodic beams with large deflections are derived using the tolerance averaging technique, cf. [7, 8].

Let $\Delta(x)=x+\Delta$, $\Omega_\Delta=\{x\in\Omega:\Delta(x)\subset\Omega\}$ be a cell with center at $x\in\Omega_\Delta$. The averaging operator for an arbitrary integrable function f is defined by:

$$<f>(x)=\frac{1}{l}\int_{\Delta(x)} f(y)dy,\quad x\in\Omega_\Delta,\quad y\in\Delta(x).\tag{2}$$

It can be shown that for periodic function f of x, its averaged value (2) is constant.

The first of the basic assumptions is *the micro-macro decomposition* of the unknown functions:

- for the transverse deflection:

$$w(x,t) = W(x,t) + h^A(x)V^A(x,t), \quad A = 1,\ldots,N, \tag{3}$$

- and for the axial displacement x:

$$u_0(x,t) = U(x,t) + g^K(x)T^K(x,t), \quad K = 1,\ldots,M, \tag{4}$$

where the functions $W(\cdot), V^A(\cdot) \in SV_d^2(\Omega,\Delta)$, $U(\cdot), T^K \in SV_d^1(\Pi,\Delta)$ are new basic unknowns, being slowly-varying functions in x; the fluctuation shape functions $h^A(\cdot) \in FS_d^2(\Omega,\Delta)$, $g^K(\cdot) \in FS_d^1(\Omega,\Delta)$ are postulated *a priori* in every problem under consideration. The new basic kinematic unknowns $W(\cdot)$ and $U(\cdot)$ are called *the macrodeflection* and *the in-plane macrodisplacements*, respectively; $V^A(\cdot)$ and $T^K(\cdot)$ are additional kinematic unknowns, called the fluctuation amplitudes.

4. The governing equations of proposed models

4.1. The governing equations of the tolerance model

After substitution the micro-macro decompositions (3) and (4) into equations (1), the next step of modelling is averaging (2) over an arbitrary periodicity cell. In case of symmetric or antisymmetric cell, some of the averaged coefficients yield zero automatically.

After some manipulations we arrive at the following system of equations:

$$\begin{aligned}
&\partial[< EA > (\partial U + \tfrac{1}{2}\partial W \partial W) + < EA\partial g^K > T^K + \\
&\qquad + \tfrac{1}{2}\underline{< EA\partial h^A \partial h^B >} V^A V^B] = 0, \\
&< EJ > \partial^4 W + < EJ\partial^2 h^A > \partial^2 V^A + < \mu > \ddot{W} + < k > W + \underline{< kh^A >} V^A + \\
&+ < c > \dot{W} + \underline{< ch^A >} \dot{V}^A - < q > + \partial(\underline{< EA\partial h^A \partial h^B >} V^A V^B \partial W) + \\
&+ [< EA > (\partial U + \tfrac{1}{2}\partial W \partial W) + < EA\partial g^K > T^K + \\
&+ \tfrac{1}{2}\underline{< EA\partial h^A \partial h^B >} V^A V^B]\partial^2 W = 0, \\
&< EJ\partial^2 h^A > \partial^2 W + < \partial^2 h^A EJ \partial^2 h^B > V^B + \underline{< \mu h^A h^B >} \ddot{V}^B + \underline{< ch^A >} \dot{W} + \\
&+ \underline{< ch^A h^B >} \dot{V}^B + \underline{< kh^A >} W + \underline{< kh^A h^B >} V^B - < qh^A > + \\
&+ \underline{< EA\partial h^A \partial h^B >} (\partial U + \tfrac{1}{2}\partial W \partial W) V^B + \underline{< EA\partial h^A \partial h^B >} V^B \partial W \partial W + \\
&+ \tfrac{1}{2}\underline{< EA\partial h^A \partial h^B \partial h^C \partial h^D >} V^B V^C V^D + \underline{< EA\partial g^L \partial h^A \partial h^B >} V^B T^L = 0, \\
&< EA\partial g^K > (\partial U + \tfrac{1}{2}\partial W \partial W) + < EA\partial g^K \partial g^L > T^L + \\
&+ \tfrac{1}{2}\underline{< EA\partial g^K \partial h^A \partial h^B >} V^A V^B = 0.
\end{aligned} \tag{5}$$

It is a system of 2+N+M differential equations for the macrodisplacements $U(\cdot)$, $W(\cdot)$ and for the fluctuation amplitudes of the deflection $V^A(\cdot)$ and of the axial displacement $T^K(\cdot)$. The coefficients of these equations are constant, some of them (the underlined ones) depend on the size l of the periodicity cell. Hence, the tolerance model describes the effect of the microstructure size on vibrations of the beams under consideration. For instance, free vibration frequencies of higher order vibrations can be analysed, which are related to the microstructure of these beams.

4.2. The governing equations of the simplified tolerance model

In order to formulate a simplified model it can be assumed that the deflection fluctuation impact on the relative elongation of the beam middle axis is negligible. Therefore, the nonlinear components of the strain that involve the fluctuation amplitudes can be omitted.

Introducing the following denotations:

$$\begin{gathered} B \equiv < EA >,\ B^K \equiv < EA\partial g^K >,\ B^{KL} \equiv < EA\partial g^K \partial g^L >,\quad Q \equiv < q >, \\ D \equiv < EJ >,\ D^A \equiv < EJ\partial^2 h^A >,\ D^{AB} \equiv < \partial^2 h^A EJ \partial^2 h^B >,\ Q^A \equiv l^{-2} < qh^A >, \\ K \equiv < k >,\ K^A \equiv l^{-2} < kh^A >,\ K^{AB} \equiv l^{-4} < kh^A h^B >,\ M \equiv < \mu >, \\ C \equiv < c >,\ C^A \equiv l^{-2} < ch^A >,\ C^{AB} \equiv l^{-4} < ch^A h^B >,\ M^{AB} \equiv l^{-4} < \mu h^A h^B >, \end{gathered} \tag{6}$$

governing equations of the simplified tolerance model take the form:

$$\begin{aligned} & \partial[B(\partial U + \tfrac{1}{2}\partial W \partial W) + B^K T^K] = 0, \\ & D\partial^4 W + D^A \partial^2 V^A + M\ddot{W} + KW + l^2 K^A V^A + C\dot{W} + l^2 C^A \dot{V}^A - Q + \\ & \quad + [B(\partial U + \tfrac{1}{2}\partial W \partial W) + B^K T^K]\partial^2 W = 0, \\ & D^A \partial^2 W + D^{AB} V^B + l^4 M^{AB} \ddot{V}^B + l^2 C^A \dot{W} + l^4 C^{AB} \dot{V}^B + l^2 K^A W + \\ & \quad + l^4 K^{AB} V^B - l^2 Q^A = 0, \\ & B^K(\partial U + \tfrac{1}{2}\partial W \partial W) + B^{KL} T^L = 0. \end{aligned} \tag{7}$$

Because the matrix B^{KL} in equation $(7)_4$ is nonsingular there exists a matrix $(B^{KL})^{-1}$ and this equation can be written as:

$$T^K = -(B^{LK})^{-1} B^L (\partial U + \tfrac{1}{2}\partial W \partial W). \tag{8}$$

Introducing the effective tensile stiffness of the beam:

$$B_0 \equiv B - B^K (B^{LK})^{-1} B^L, \tag{9}$$

denoting:

$$\overline{N} = B_0\left(\partial U + \tfrac{1}{2}\partial W \partial W\right), \tag{10}$$

and after substituting the right-hand side of (8) into $(7)_1$ we have, instead of (7), the following equations:

$$\begin{aligned}
&\partial\overline{N} = 0,\\
&D\partial^4 W + D^A \partial^2 V^A + M\ddot{W} + KW + l^2 K^A V^A + C\dot{W} + l^2 C^A \dot{V}^A -\\
&- Q + \overline{N}\partial^2 W = 0,\\
&D^A \partial^2 W + D^{AB} V^B + l^4 M^{AB} \ddot{V}^B + l^2 C^A \dot{W} + l^4 C^{AB} \dot{V}^B + l^2 K^A W +\\
&+ l^4 K^{AB} V^B - l^2 Q^A = 0,\\
&T^K = -(B^{LK})^{-1} B^L (\partial U + \tfrac{1}{2}\partial W \partial W).
\end{aligned} \tag{11}$$

Equations (11) stand the system of differential-algebraic equations. Similarly to equations (7) the above equations have the terms dependent of the microstructure parameter *l*. Hence, the simplified tolerance model makes it possible also to investigate the effect of the microstructure size on vibrations of these beams.

4.3. The governing equations of the asymptotic model

Neglecting in equations (7) or (11) the terms with the microstructure parameter *l* and introducing the effective stiffness of bending of beam:

$$D_0 \equiv D - D^A (D^{AB})^{-1} D^B, \tag{12}$$

we arrive at the equations in the form:

$$\begin{aligned}
&\partial\overline{N} = 0, \quad \overline{N} = B_0(\partial U + \tfrac{1}{2}\partial W \partial W),\\
&D_0 \partial^4 W + M\ddot{W} + KW + C\dot{W} - Q + \overline{N}\partial^2 W = 0,\\
&V^A = -(D^{AB})^{-1} D^B \partial^2 W,\\
&T^K = -(B^{LK})^{-1} B^L (\partial U + \tfrac{1}{2}\partial W \partial W).
\end{aligned} \tag{13}$$

The above equations do not describe the effect of the microstructure size on the behaviour of the periodic beams under consideration. Hence, the asymptotic model makes it possible to analyse vibrations on the macrolevel only.

5. Remarks

In this contribution the mathematical model, called *the tolerance model*, is shown, which describes dynamics of a periodically nonhomogeneous beam. The governing equations of this model are obtained by using the tolerance method, cf. [9, 8, 7]. Hence, the fundamental equations with highly oscillating, periodic, noncontinuous functional coefficients are replaced by the equations with constant coefficients.

The following general remarks can be formulated.

1. It can be observed that only *the tolerance model* and *the simplified tolerance model* make it possible to investigate *the effect of the microstructure size* on dynamic problems of periodic beams under consideration, e.g. the "higher order" vibrations related to the beam microstructure.
2. The governing equations of both *the tolerance models* have a physical sense for unknowns W, U, V^A, A=1,...,N, T^K, K=1,...,M, being slowly-varying functions.
3. *The asymptotic model* of periodic beams makes it possible to investigate only lower order (fundamental) vibrations.

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