Vibrations in Physical Systems Vol.26 (2014)

Modal Analysis of Viscous Flow and Reduced Order Models

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Abstract

Phenomena occurring in the flows are very complex. Their interpretation, as well as an effective impact on them in the flow control is often only possible with the use of modal analysis and low-dimensional models. In this paper, the selected modal decomposition techniques, namely Proper Orthogonal Decomposition (POD), Dynamic Mode Decomposition (DMD) and global stability analysis, are briefly introduced. The design of Reduced Order models basing on Galerkin projection is presented on the example of the flow past a bluff body. Finally, the issues of widening of the application of the models are addressed.

Keywords: Reduced Order Models, ROM, Galerkin method, modal analysis, POD, DMD, stability

1. Introduction

The progress in the aerospace and automotive industry is possible by improvement of newly-designed vehicles. The reduction of aerodynamic drag, generated noise and exhaust fumes emission, as well as the increase of lift and the growth of performance might be achieved by the change in the flow phenomena, that might be obtained by the means of flow control. Such control is expected to be the most effective when the operation of the actuators is in accordance with the state/phase of the flow, measured by the sensors and processed by the controller with proper model of the flow.

The high-fidelity solution based on Navier-Stokes, LES/DES, RANS or even Euler equations, is very time-consuming. Real-life problems consisting of millions of degrees of freedom are possible to be solved only on parallel machines like computer clusters. Due to the high computational complexity of the high-fidelity flow models, an essential element of the closed-loop controller is low dimensional model of the flow. Such a model strongly depends on the proper choice of modal basis used in the approximation and projection stages.

In this paper, a short overview of the methods of modal analysis of viscous flows, described by incompressible Navier-Stokes equations (1) is presented, and a design of Reduced Order Models of the flow basing on the modal decomposition is described.

$$\dot{V} + \nabla(V \otimes V) + \nabla P - \tfrac{1}{\mathrm{Re}} \Delta V = 0 \tag{1}$$

Finally, the methods of the design of broadband Galerkin models, capable to cover wide range of operating and boundary conditions, are briefly discussed.

2. Modal decomposition techniques

The most popular method of modal analysis is Proper Orthogonal Decomposition (POD) [1,2]. In this method, the M flow vectors (snapshots), resulting from experiment or numerical analysis, are centred using time-averaged solution U_0:

$$V_i' = V_i - U_0 , \quad \text{where} \quad U_0 = \frac{1}{M} \sum_{i=1}^{M} V_i \tag{2}$$

Next, the auto-correlation matrix for the matrix containing fluctuation vectors is calculated:

$$C = \frac{1}{M} S S^T , \quad \text{where} \quad S = [V_1', V_2', \ldots, V_M'] \tag{3}$$

The eigenvectors of the auto-correlation matrix (POD modes, fig. 1), related to the eigenvalues of the largest module, might be used in Reduced Order Modelling of the flow.

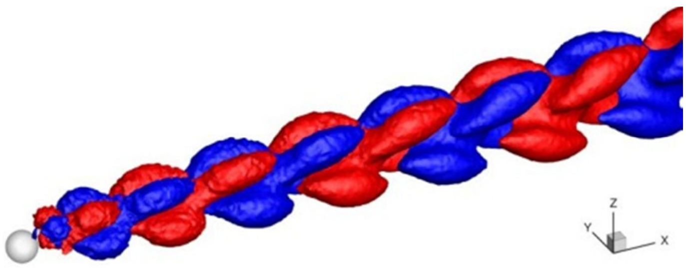

Figure 1. The most dominant POD mode for a flow past a sphere. Iso-surfaces of transverse velocity V_y are depicted

Another, physical mode basis results from global, linear stability analysis of the flow [3,4]. The decomposition of the instantaneous flow field onto base (steady or time-averaged) solution $\overline{V}$ and small, oscillatory disturbance:

$$V' = \widetilde{V} e^{-\lambda t} \tag{4}$$

and the linearization of the resulting equation leads to the generalized eigenvalue problem:

$$\lambda \widetilde{V} + \nabla(\widetilde{V} \otimes \overline{V}) + \nabla(\overline{V} \otimes \widetilde{V}) + \nabla \widetilde{P} - \tfrac{1}{\mathrm{Re}} \Delta \widetilde{V} = 0 \tag{5}$$

Complex eigenvectors of such problem (global stability eigenmodes, Fig. 2) represent the behaviour of the dynamical system close to fixed point, describing, for example, the transition between symmetrical wake and von Karman street of vortices in the case of bluff body wakes.

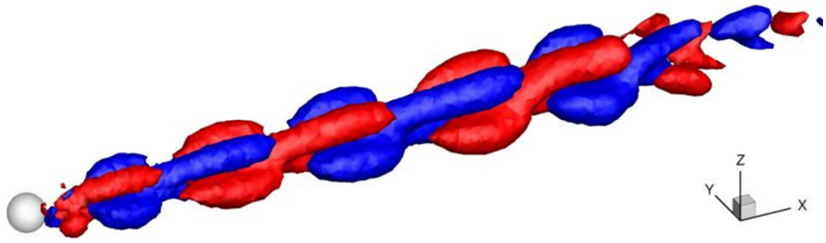

Figure 2. Real part of the most dominant eigenmode for a flow past a sphere. Isosurfaces of transverse velocity V_y are depicted

The two aforementioned methods for modal basis design have very important drawbacks. For example, as will be discussed in further section of this paper, POD modes represent very narrow range of the conditions of the flow. Additionally, there might be problems to obtain correct modes outside the limit cycle, when the oscillation is amplified or suppressed. On the other hand, eigenmodes of global stability analysis are very difficult to obtain, particularly for 3D flows.

To overcome these problems, the idea of Dynamic Mode Decomposition [5,6] has been proposed. In DMD, it is assumed that any instantaneous solution might be obtained from a linear combination of previous solutions:

$$q(t+\Delta t) \approx e^{\Delta t A}\, q(t) \tag{6}$$

The product of previous state vectors and a linear operator $\widetilde{A} \approx e^{\Delta t A}$ might be approximated using the product:

$$\widetilde{A}\, V_{0\ldots n} \approx V_{0\ldots n}\, S, \tag{7}$$

where $V_{0..n}$ is the sequence of known solutions and S is the companion matrix as defined below:

$$S = \begin{pmatrix} 0 & 0 & \ldots & 0 & c_0 \\ 1 & 0 & \ldots & 0 & c_1 \\ 0 & 1 & \ldots & 0 & c_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & c_n \end{pmatrix} \tag{8}$$

The coefficients $c_0 \ldots c_n$ are obtained from the solution of the over-determined system of equations (1). The eigenvectors of matrix S are used to obtain the DMD modes (Fig. 3), while the eigenvalues determine modal growth ratios (real part) and frequencies (imaginary parts).

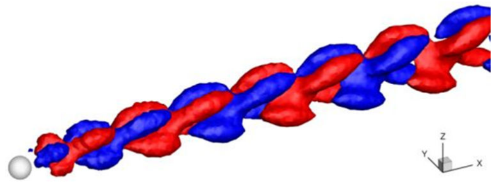

Figure 3. Real part of the most dominant DMD mode for a flow past a sphere. Iso-surfaces of transverse velocity V_y are depicted

3. Reduced Order Models

Reduction of dimension of the flow model is based on the assumption that the velocity field might be decomposed onto the sum of the base flow U_0 and n products of spatial modes U_j and temporal coefficients a_j (9):

$$V(t_i) \approx U_0 + \sum_{j=1}^{n} a_j(t_i)U_j = V^{[n]}(t_i) \tag{9}$$

Such decomposition leads to approximated governing equation. Truncation of mode basis to a limited (preferably small) number of modes results in the residuum:

$$\dot{V}^{[n]} + \nabla\left(V^{[n]} \otimes V^{[n]}\right) + \nabla P^{[n]} - \tfrac{1}{\mathrm{Re}}\Delta V^{[n]} = R^{[n]} \tag{10}$$

Projection of the residuum onto the space spanned by the modes, called Galerkin projection [7], is equivalent in Hilbert space to the zeroing of the integrals of the products of modes and the residuum:

$$\left(U_i, R^{[n]}\right)_\Omega = \int_\Omega U_i R^{[n]} d\Omega = 0 \tag{11}$$

The result of this projection is a system of ODEs, called Galerkin system:

$$\dot{a}_i = \frac{1}{\mathrm{Re}}\sum_{j=0}^{N} l_{ij} a_j + \sum_{j=0}^{N}\sum_{k=0}^{N} q_{ijk} a_j a_k \,, \tag{12}$$

where linear and quadratic terms are derived as follows:

$$l_{ij} = \left(U_i, \Delta U_j\right)_\Omega \quad \text{and} \quad q_{ijk} = -\left(U_i, \nabla \cdot \left(U_j \otimes U_k\right)\right)_\Omega \,. \tag{13}$$

The proper choice of mode basis makes the resulting model comparable to the high-fidelity data from Direct Numerical Simulation (DNS) of Navier-Stokes equations.

4. Enhanced Galerkin models

Reduction of fluid model by projection of governing equations onto orthonormal mode basis (Galerkin Projection) results in approximate flow solution. Truncation of mode basis, as well as differences between low-dimensional model formulation and high-dimensional data used in mode expansion (like the neglection of pressure term, assumption of incompressibility, etc.), result in the deterioration of model quality.

To improve the quality of the Reduced Order Models of fluid flow, corrections to the linear and quadratic terms might be added, as computed in Genetic Algorithm-based calibration [8]:

$$\dot{a}_i = \frac{1}{\mathrm{Re}} \sum_{j=0}^{N} \left(l_{ij} + l_{ij}^{+} \right) a_j + \sum_{j=0}^{N} \sum_{k=0}^{N} \left(q_{ijk} + q_{ijk}^{+} \right) a_j a_k \tag{14}$$

Mode basis used in Galerkin expansion allows reconstruction of the flow for a given set boundary and operating conditions. In the case of changing flow conditions, used mode basis has to be adjusted, for example using hybrid models [7] designed with the basis consisting of both empirical and physical modes, or mode parameterization, for example with some kind of look-up table [9] or Double-POD [10] approach. Another choice is Continuous Mode Interpolation [11], where the mode bases for two or more operating/boundary conditions are interpolated by referring to the Fredholm eigen-problem in space domain:

$$\int_{\Omega} A(x,y) U_i(y) dy = \lambda U_i(x), \tag{15}$$

with autocorrelation function (kernel) A:

$$A^{\kappa}(x,y) = U_1^{\kappa}(x) \otimes U_1^{\kappa}(y) + U_2^{\kappa}(x) \otimes U_2^{\kappa}(y) \tag{16}$$

In the case of interpolation between two states, the Fredholm kernel is linearly interpolated in $\kappa \in [0;1]$:

$$A^{\kappa} = A^0 + \kappa \left(A^1 - A^0 \right) \tag{17}$$

The approach presented above allows smooth and continuous interpolation between corresponding structures (modes) for different operating conditions, enabling e.g. the modelling of the transition from fixed point dynamics to limit-cycle oscillations (Fig. 4).

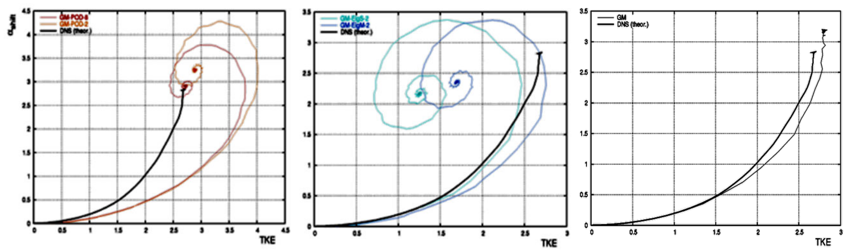

Figure 4. Phase portraits of Galerkin models, compared to reference data (thick black line): empirical POD-based (left), stability-based models (center) and the model using Continuous Mode Interpolation (right) for a flow past a NACA-0012 airfoil [12]

5. Conclusions

Modal analysis of the flow and its Reduced Order Models are key enablers for feedback flow control. In this paper basic issues related to the model order reduction and Galerkin projection are presented and the most popular methods for obtaining the mode basis, such as POD, global stability analysis and DMD, are described. Modelling of the flow in changing boundary and operating conditions is possible with the use of parameterization methods such as Continuous Mode Interpolation.

Acknowledgments

This work has been funded by National Centre of Science under research grant no. DEC-2011/01/B/ST8/07264.

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