Vibrations in Physical Systems Vol.26 (2014)

Nonlinear Vibrations of Rotating System Near Resonance

Roman STAROSTA
Poznan University of Technology, Institute of Applied Mechanics
ul. Piotrowo 3, 60-965 Poznań, Poland
roman.starosta@put.poznan.pl

Grażyna SYPNIEWSKA-KAMIŃSKA
Poznan University of Technology, Institute of Applied Mechanics
ul. Piotrowo 3, 60-965 Poznań, Poland
grazyna.sypniewska-kaminska@put.poznan.pl

Abstract

The paper concerns analysis of nonlinear vibration of the rotating system consisted of two disks and shaft. The analytical multiple time scale method is applied to the analysis dynamics of the system near main resonance. The transition phenomenon depending on the value of the nonlinearity parameter is discussed. All the analytical results have been confirmed numerically.

Keywords: nonlinear vibrations, asymptotic methods, resonance

1. Introduction

Torsional vibrations are one of main problem in design of the power transmission systems [2]. Dynamic stresses caused by torsional vibrations, especially when their amplitudes grow significantly near resonance, may be very large and lead to failure of the whole system.

Both discrete and continuous models are commonly used in order to investigate the torsional vibrations of the power transmission systems [1,4,6]. We have attempted to apply the Limiting Phase Trajectories (LPT) method in order to investigate nonlinear torsional vibrations. LPT is an analytical approximate method, developed recently by Manevitch and used to analyse of discrete systems [3]. The discrete model was chosen as the most convenient to use this method. Similar approach was applied in [5].

2. Mathematical model

Let us consider a rotating system, consisting of two disks mounted on a shaft. The system studied is shown in Fig. 1. The disks are considered as rigid. Their moments of inertia around the axis of rotation are denoted by I_1 and I_2, respectively. The shaft is relatively thin and light, so its mass may be neglected. The shaft provides torsional stiffness only. The nonlinear relationship between the angle of twist and the torque was assumed. Two coefficients of stiffness, marked by k and k_n, are introduced. Moreover viscous damping, of which the damping coefficient equal to c, is taken into account. The whole system is mounted on frictionless bearings which are also ideal in the geometric sense. One of the disks is under the action of the harmonically changing torque

$M(t) = M_0 \cos(p_0 t)$. The system has two degree of freedom. The angles of rotation of both wheels are chosen as the generalized coordinate.

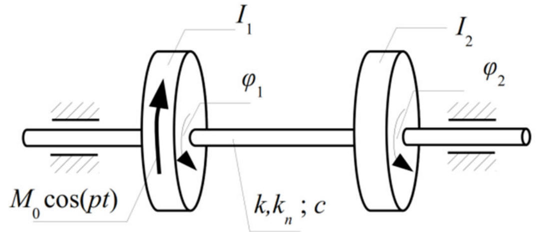

Figure 1. Model of rotating system

The Lagrangian of the system is as follows:

$$L = \frac{1}{2} I_1 \dot{\varphi}_1^2 + \frac{1}{2} I_2 \dot{\varphi}_2^2 - \frac{1}{2} k (\varphi_1 - \varphi_2)^2 - \frac{1}{4} k_n (\varphi_2 - \varphi_1)^4 , \quad (1)$$

where I_1 and I_2 are the moments of inertia, k and k_n are the stiffness coefficients.

The equations of motion are as follows

$$I_1 \ddot{\varphi}_1 - k(\varphi_2 - \varphi_1) - k_n (\varphi_2 - \varphi_1)^3 - c(\dot{\varphi}_2 - \dot{\varphi}_1) = M_0 \cos(p_0 t), \quad (2)$$

$$I_2 \ddot{\varphi}_2 - k(\varphi_2 - \varphi_1) - k_n (\varphi_2 - \varphi_1)^3 - c(\dot{\varphi}_2 - \dot{\varphi}_1) = 0 . \quad (3)$$

Dividing eqs. (2) and (3) respectively by I_1 and I_2 and substracting then eq (2) from eq (3) we obtain the equation, in which only the differences of the unknown functions and their derivatives are present. Hence let us introduce the new function

$$\Phi(t) = \varphi_1(t) - \varphi_2(t) . \quad (4)$$

The function $\Phi(t)$ is simply the angle of twist of the shaft between the discs. The substitution (4) leads to the equation

$$\ddot{\Phi}(t) + K^2 \Phi(t) + K_n \Phi(t)^3 + C \dot{\Phi}(t) = \frac{M_0}{I_1} \cos p_0 t , \quad (5)$$

where $K = \sqrt{k / I_1 + k / I_2}$, $\eta_e = k_n / I_1 + k_n / I_2$, $C = c / I_1 + c / I_2$.

The eq. (5) describes the internal motion which is especially important with respect of vibrational process.

Introducing dimensionless time $\tau = K t$, the more convenient form of the equation of motion can be written:

$$\ddot{\phi}(\tau) + \phi(\tau) + \eta_e \phi(\tau)^3 + \gamma_e \dot{\phi}(\tau) = \mu \cos p\tau , \quad (6)$$

where $\mu = M_0 / I_1 K^2$, $p = p_0 / K$, $\gamma_e = C / K$ and ϕ is the angle of twist with respect to τ .

3. Asymptotic solution

Further analysis concerns the eq. (6). Let us assume that the system is weakly nonlinear and moreover the damping coefficient and the amplitude of the external torque are of the order of small parameter. The above assumptions allow to write the governing equation in the form:

$$\ddot{\phi}+2\varepsilon\gamma\dot{\phi}+\phi+8\varepsilon\eta\phi^3=2\varepsilon f\cos(p\tau), \tag{7}$$

where $2\varepsilon\gamma=\gamma_e$, $8\varepsilon\eta=\eta_e$, $2\varepsilon f=\mu$.

The initial conditions are assuming to be homogeneous $\phi(0)=0$, $\dot{\phi}(0)=0$.

Let us introduce the function $v(\tau)=\dot{\phi}(\tau)$. Then the eq. (7) can be written as the set of equations of the first order:

$$\begin{aligned} v-\dot{\phi}&=0,\\ \dot{v}+2\varepsilon\gamma\dot{\phi}+\phi+8\varepsilon\eta\phi^3&=2\varepsilon f\cos(p\tau). \end{aligned} \tag{8}$$

The key point for the next analysis is the introduction complex functions

$$\psi=v+i\phi \text{ and } \overline{\psi}=v-i\phi. \tag{9}$$

Substituting the definitions (9) into eq. (8) we obtain the equation

$$\frac{d\psi}{d\tau}-i\psi+\gamma\varepsilon(\psi+\overline{\psi})+i\eta\varepsilon^2(\psi-\overline{\psi})^3=2\varepsilon f\cos(p\tau), \tag{10}$$

with the initial conditions $\psi(0)=0$, which is equivalent to the system (8).
After introducing once more substitution

$$\psi=\Psi e^{i\tau} \text{ and } \overline{\psi}=\overline{\Psi}e^{-i\tau}, \tag{11}$$

\we get the following equation with unknown complex function $\Psi(\tau)$.

$$\begin{aligned} &\frac{d\Psi}{d\tau}+\gamma\varepsilon\left(\Psi+\overline{\Psi}e^{-2i\tau}\right)+i\eta\varepsilon\left(\Psi^3e^{2i\tau}-\overline{\Psi}^3e^{-4i\tau}-3|\Psi|^2\Psi+3|\Psi|^2\overline{\Psi}e^{-2i\tau}\right)=\\ &=2\varepsilon e^{-i\tau}f\cos(p\tau), \end{aligned} \tag{12}$$

with the initial condition $\Psi(0)=0$. The appropriate complex conjugate formulation could be written as well.

Let us focus attention on the case of the main resonance, that occurs when $p\approx 1$. In order to consider this case, the small detuning parameter σ is introduced in the form $p=1+\sigma=1+\varepsilon\tilde{\sigma}$.

The initial value problem (12) is solved with the help of the Multiple Scale Method. Let us introduce two time scales $\tau_0=\tau$ and $\tau_1=\varepsilon\tau$. The assumed form of the solution is as follows:

$$\Psi(\tau)=\Psi_0(\tau_0,\tau_1)+\varepsilon\,\Psi_1(\tau_0,\tau_1)\,. \tag{13}$$

After substituting (13) into eq. (12) and arranging it with respect to powers of small parameter ε we obtain

- the equation of order ε^0

$$\frac{\partial\Psi_0}{\partial\tau_0}=0\,, \tag{14}$$

- the equation of order ε^1

$$\begin{gathered}\frac{\partial\Psi_1}{\partial\tau_0}+\frac{\partial\Psi_0}{\partial\tau_1}+\gamma\left(\Psi_0+\overline{\Psi}_0 e^{-2i\tau_0}\right)+\\ +i\eta\left(\Psi_0^3 e^{2i\tau_0}-3\left|\Psi_0\right|^2\Psi_0+3\Psi_0^2\overline{\Psi}_0 e^{-2i\tau_0}-\overline{\Psi}_0^3 e^{-4i\tau_0}\right)=\\ =f\left(e^{i(\tilde{\sigma}\tau_1)}+e^{-i(2\tau_0+\tilde{\sigma}\tau_1)}\right),\end{gathered} \tag{15}$$

From eq. (14) appears that $\Psi_0=\Psi_0(\tau_1)$.

The solution of eq. (15) should be limited. In that reason, the secular terms in (15) should be eliminated. That leads to the solvability condition

$$\frac{\partial\Psi_0}{\partial\tau_1}+\gamma\Psi_0-3i\eta\left|\Psi_0\right|^2\Psi_0=fe^{i(\tilde{\sigma}\tau_1)}\,. \tag{16}$$

Introducing polar representation

$$\Psi_0(\tau_1)=a(\tau_1)e^{i\delta(\tau_1)}, \tag{17}$$

where $a,\delta\in\Re$, we obtain the new form of the solvability condition

$$\frac{da}{d\tau_1}+ia\frac{d\delta}{d\tau_1}+\gamma a-3i\eta a^3=fe^{i(\tilde{\sigma}\tau_1)}e^{-i\delta}\,. \tag{18}$$

Taking advantage of the fact that $a(\tau_1)$ and $\delta(\tau_1)$, then multiplying the eq. (18) by ε and returning to the original denotations occurring in (6), one can obtain

$$\frac{da}{d\tau}+ia\frac{d\delta}{d\tau}+\frac{1}{2}\gamma_e a-\frac{3}{8}i\eta_e a^3=\frac{\mu}{2}e^{i(\sigma\tau)}e^{-i\delta}\,. \tag{19}$$

Writing the exponential functions in the trigonometric form, and then separating real and imaginary parts in the equation, we have

$$\begin{gathered}\frac{da}{d\tau}+\frac{1}{2}\gamma_e a=\frac{\mu}{2}\cos\theta,\\ -a\frac{d\theta}{d\tau}+a\sigma-\frac{3}{8}\eta_e a^3=\frac{\mu}{2}\sin\theta,\end{gathered} \tag{20}$$

where $\theta=\sigma\tau-\delta$ is modified phase. The eqs. (20) describe the modulation of the amplitude a and the modified phase θ.

4. Non-steady vibrations

In order to apply the LPT method, let us consider the non-damped vibrations ($\gamma_e = 0$). In that case the set of equations (20) has the first integral

$$H = -a\frac{\mu}{2}\sin\theta + \sigma\frac{a^2}{2} - \frac{3}{32}\eta_e a^4 = \text{const}\,, \tag{21}$$

where the constant of the right side depends on the initial conditions. The eq. (21) represents one-parameter family of the curves on the plane (a, θ).

We are especially interested in the case, when maximal energy exchange between the system and the external loading appears. This situation takes place for H=0. In that case the first integral (21) has the form:

$$-16\mu\sin\theta + 16\sigma a - 3\eta_e a^3 = 0\,. \tag{22}$$

It is easy to show that the curve given by eq. (22) has extrema for $\theta = -\pi/2$ and $\theta = \pi/2$. The qualitative change in the behaviour of the system is observable for the critical value of nonlinearity parameter

$$\eta_e = \eta_c = \frac{64\sigma^3}{81\mu^2}\,. \tag{23}$$

In Figure 2 the trajectory curves on the plane (a, θ) for three values of η_e obtained from (22) are presented. All the graphs presented in the Figs. 2-6 are made assuming σ = 0.01, μ= 0.002.

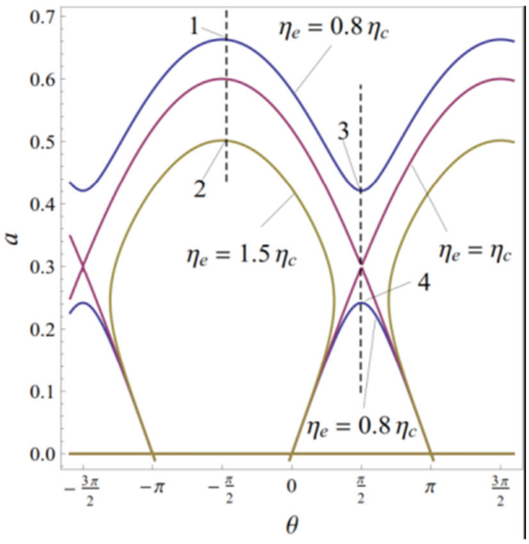

Figure 2. Phase trajectories for three values of η_e.
Points 1, 2, 3, 4 identify roots of the eq. (22) for $\theta = -\pi/2$ and $\theta = \pi/2$

The maximum values of the amplitude $a_{max}(\eta_e)$ of vibrations are presented in Fig. 3. Points 1,2,3 and 4 in this figure identify the same solutions as in Fig. 2.

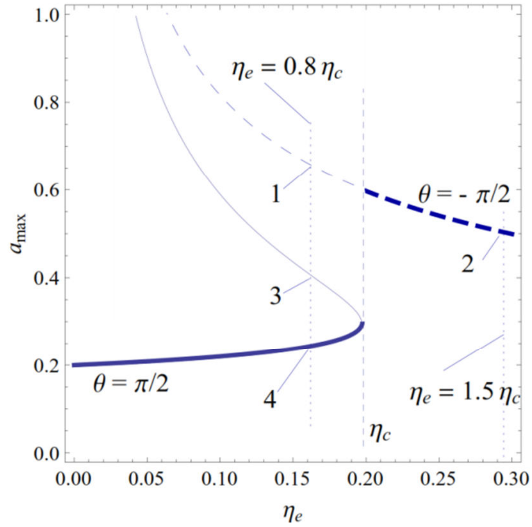

Figure 3. Graphs of $a_{max}(\eta_e)$ according to (22); thick line identifes a_{max}; thin line reflect the open trajectories in the plane (a, θ) and do not describe vibrations

There is one more qualitative change in the phase portrait of vibrations on the plane (a, θ) for $\eta_e = 2\eta_c$. The metamorphoses of behavior of the system is clearly visible in time history of general co-ordinate. When η_e exceeds the critical value η_c or $2\eta_c$, the shape of modulation of amplitude rapidly changes. Amplitude modulation in time τ, obtained from the eqs. (20), are presented in Figs. 4-6.

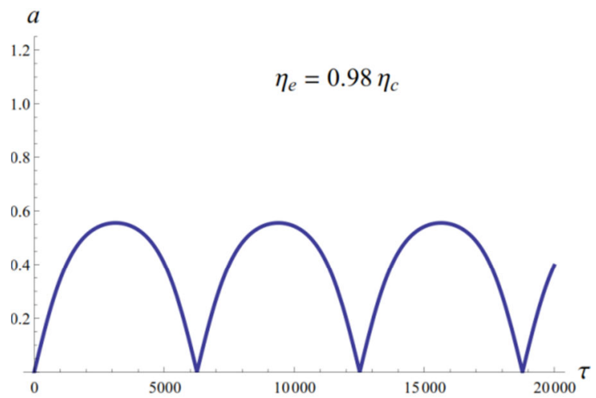

Figure 4. Amplitude modulation for η_e just below η_c $(\sigma = 0.01, \mu = 0.002)$

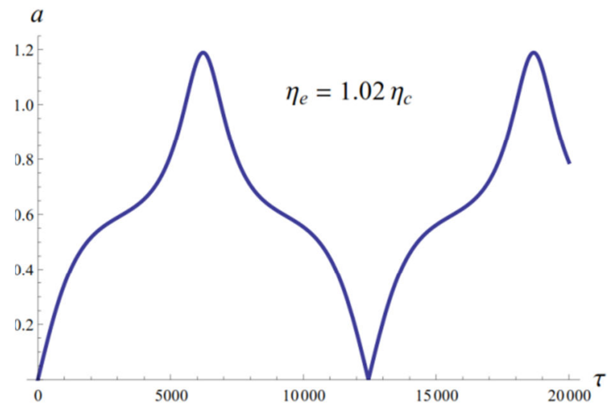

Figure 5. Amplitude modulation for η_e just above η_c ($\sigma = 0.01, \mu = 0.002$)

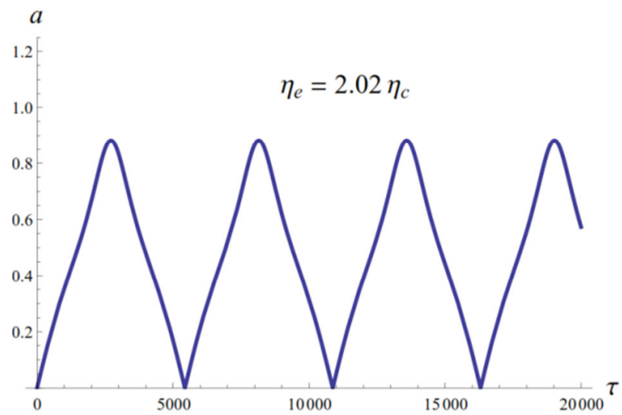

Figure 6. Amplitude modulation for η_e just above $2\,\eta_c$ ($\sigma = 0.01, \mu = 0.002$)

There are no other transformations of the phase-plane portraits and qualitative changes in behavior of a and θ for $\eta_e > 2\,\eta_c$.

5. Conclusions

Analysis of the nonlinear disks-shaft system has been done. The asymptotical Multiple Time Scale method has been adopted to solve the problem. It allowed us to exhibit an important dynamical transition in the non-steady state vibrations, leading to the drastic change of amplitude with increasing nonlinearity parameter (see Figs. 4- 6). The maximal amplitudes and trajectories on the plane (a, θ) have been presented (see Figs. 2,3). The graphs presented in the paper indicate the intensive energy exchange between the system and external excitation. All presented results have been obtained analytically and confirmed numerically.

Acknowledgments

The paper is supported by grant DS 21-418/2014 DS-PB.

References

1. Y.Huang Ying, S.Yang, F.Zhang, C.Zhao, Q.Ling, H.Wang, *Non-linear Torsional Vibration Characteristics of an Internal Combustion Engine Crankshaft Assembly*, Chinese J. Mech. Eng., **25**(4) (2012) 797- 808.
2. J. Kožešnik, Dynamika maszyn, WNT 1963.
3. L.I. Manevitch, A.I. Musienko, *Limiting Phase Trajectories and Energy Exchange Between Anharmonic Oscillator and External Force*, Nonlinear Dynamics **58** (2009) 633- 642.
4. B. Porter, *Non-Linear Torsional Vibration Of A Two-Degree of-Freedom System Having Variable Inertia*, J. Mech. Eng. Sci., **7**(1) (1965), 101- 113.
5. R. Starosta, *Nieliniowa dynamika układów dyskretnych w obszarach rezonansów w ujęciu asymptotycznym,* Nr 457 Rozprawy, Wydawnictwo PP, Poznań 2011.
6. Z. Wenming, W. Bohua, Z. Shuangshuang, L. Shuang, *Study on Bifurcation and Chaotic Motion of a Strongly Nonlinear Torsional Vibration System under Combination Harmonic Excitations*, Int. J. Com. Sci. **10**(3) (2013) 293- 299.