Approximate Method for Determination of Dynamic Characteristics of Structures with Viscoelastic Dampers

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Abstract

An approximate method for determination of dynamic characteristics of structures with viscoelastic dampers is proposed in this paper. A fractional derivative is used to describe the dynamic behaviour of viscoelastic dampers. The method is based on a continuous dependency of the sensitivity of eigenvalue on a certain artificially introduced parameter which scaled up the influence of the damping term in the eigenvalue problem. Some results of a representative calculation are also presented and briefly discussed.

Keywords: structures, viscoelastic dampers, generalized fractional model, dynamic characteristics

1. Introduction

Natural frequencies, non-dimensional damping ratios and modes of vibration are the fundamental dynamic characteristics of every structural system. These quantities are obtained after solving appropriately defined eigenvalue problems. It is a well known procedure when the damping of systems can be neglected or when the so-called proportional damping could be assumed. The problem is much more complex when damping takes place because the eigenvalue problem is often nonlinear and because complex calculations are involved. The procedure of determination of dynamic characteristics is even more complicated when the fractional derivative modes are used to describe viscoelastic (VE) dampers. In this case, usually, an advanced procedure, called the continuation method, is used to solve the nonlinear eigenvalue problem [1, 2]. Adhikari [3] used the Neumann expansion method to obtain first and second order approximations for complex eigenvectors.

In this paper, the method of determination of an approximate solution to the nonlinear eigenvalue problem describing the dynamic properties of structures with fractional dampers is presented. The method used a solution to the classical eigenvalue problem without damping and a differential equation to calculate the natural frequencies and non-dimensional damping ratios sought. Only a partial solution to the classic eigenvalue problem is needed. The method presented is an extension of the method recently proposed by Lazaro [4] but, in contrast to that method, only a partial solution to the classic eigenvalue problem is necessary and the method is extended to the case of a system of which the viscoelastic properties of dampers or materials are described by fractional derivatives. A previous approach in a similar direction was presented, also by Lazaro, in [5].

2. Equation of motion of structures with viscoelastic (VE) dampers

The elastic, planar frame structures with VE dampers are considered. The fractional model, shown in Fig. 1, is used as a model of dampers. It consists of the fractional Kelvin element which is connected in parallel with the fractional Maxwell element. The rhombus shown in the figure denotes the viscoelastic or springpot element [6]. This model of damper can be regarded as a generalized one. A set of specific models arise from it: the simple fractional Maxwell (when $k_0 = c_0 = 0$), the fractional Kelvin model (when $k_1 = c_1 = 0$) and the fractional Zener model (when $c_0 = 0$). This means that almost all of the fractional models known in the literature up to now are taken into account by the above fractional model. Here k_0, k_1 and c_0, c_1 are the stiffness and damping factors of damper, respectively, and α is the order of the fractional derivative; ($0 < \alpha \le 1$). Well known classic rheological models of damper are obtained for $\alpha = 1$.

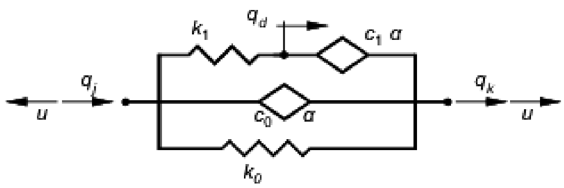

Figure 1. Mechanical diagram of the fractional model of damper

The total force in this model, $u(t) = u_0(t) + u_1(t)$, is the sum of forces that occur in the Kelvin element $u_0(t)$ and the force in the fractional Maxwell element $u_1(t)$, i.e.:

$$u_0(t) = k_0(q_k(t) - q_j(t)) + c_0 D_t^\alpha (q_k(t) - q_j(t)), \tag{1}$$

$$u_{1s}(t) = k_1(q_d(t) - q_j(t)), \qquad u_{1d}(t) = c_1 D_t^\alpha (q_k - q_d(t)), \tag{2}$$

where the symbol $D_t^\alpha(\bullet)$ denotes the Caputo or Riemann-Liouville fractional derivative of $(\bullet)$ of the order α with respect to time t. The symbol $q_d(t)$ denotes the so-called "internal variable" (see also [6, 7]). It is easy to find that $u_{1s}(t) = u_{1d}(t) = u_1(t)$.

The equation of motion of structures with VE dampers could be written in the following form (see also [6, 7]):

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}D_t^\alpha \mathbf{q}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{P}(t) \tag{3}$$

Here, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are the $(n \times n)$ global mass, damping and stiffness matrices, respectively. $\mathbf{P}(t)$ is the vector of excitation forces and $\mathbf{q}(t)$ is the $(n \times 1)$ global vector of displacements, which contains also all internal variables $q_d(t)$. For the sake of simplicity, the damping properties of structure are neglected. The mass and damping matrices are often singular and the stiffness matrix is positively defined.

Assuming that $\mathbf{P}(t) = \mathbf{0}$ and applying the Laplace transform (with zero initial conditions), the following nonlinear eigenvalue problem is obtained from Eq (3):

$$(s^2\mathbf{M}+s^\alpha\mathbf{C}+\mathbf{K})\overline{\mathbf{q}}=\mathbf{0} \tag{4}$$

where s is the Laplace variable and $\overline{\mathbf{q}}$ is the Laplace transform of $\mathbf{q}(t)$.

3. Approximate method of solution to the nonlinear eigenvalue problem

First of all, the artificial parameter p is introduced and Eq. (4) is rewritten as:

$$\mathbf{D}(s(p),\, p)\overline{\mathbf{q}}(p)=(s^2\mathbf{M}+s^\alpha p\mathbf{C}+\mathbf{K})\overline{\mathbf{q}}=\mathbf{0} \tag{5}$$

For $p=1$, the solution to the eigenvalue problem (4) is obtained whereas for $p=0$, Eq. 5 is reduced to the following linear eigenvalue problem

$$(s^2\mathbf{M}+\mathbf{K})\overline{\mathbf{q}}=\mathbf{0} \tag{6}$$

which has a well known set of solutions of the type $s=\mathrm{i}\omega$, $\overline{\mathbf{q}}=\mathbf{a}$, where ω and $\mathbf{a}$ are the natural frequency and mode of vibration, respectively, and $\mathrm{i}=\sqrt{-1}$. Let us note that the influence of the damper's stiffness is still incorporated in the stiffness matrix $\mathbf{K}$. It is assumed that the eigenvector $\overline{\mathbf{q}}$ fulfills the following normalization condition:

$$\overline{\mathbf{q}}^T(p)[2s(p)\mathbf{M}+\alpha s^{\alpha-1}p\mathbf{C}]\overline{\mathbf{q}}(p)=1 \tag{7}$$

Now, the sensitivity of the solution to the eigenvalue problem with respect to changes of parameter p will be analyzed. After differentiating Eqs (5) and (7) with respect to parameter p, the following set of equations are obtained (see also [7]):

$$(s^2\mathbf{M}+s^\alpha p\mathbf{C}+\mathbf{K})\frac{\partial\overline{\mathbf{q}}}{\partial p}+(2s\mathbf{M}+\alpha s^{\alpha-1}p\mathbf{C})\,\overline{\mathbf{q}}\,\frac{\partial s}{\partial p}=-s^\alpha\mathbf{C}\,\overline{\mathbf{q}} \tag{8}$$

$$\overline{\mathbf{q}}^T(2s\mathbf{M}+\alpha s^{\alpha-1}p\mathbf{C})\frac{\partial\overline{\mathbf{q}}}{\partial p}+\frac{1}{2}\overline{\mathbf{q}}^T[2\mathbf{M}+\alpha(\alpha-1)s^{\alpha-2}p\mathbf{C}]\overline{\mathbf{q}}\frac{\partial s}{\partial p}=-\frac{1}{2}\alpha s^{\alpha-1}\overline{\mathbf{q}}^T\mathbf{C}\overline{\mathbf{q}} \tag{9}$$

from which the sensitivity of the eigenvector $\partial\overline{\mathbf{q}}/\partial p$ and the sensitivity of the eigenvalue $\partial s/\partial p$ can be found.

Equation (8) is multiplied by $\overline{\mathbf{q}}^T$ and transformed to the following form:

$$\frac{\partial s}{\partial p}=-s^\alpha(p)\frac{A(\overline{\mathbf{q}}(p))}{B(s(p),\overline{\mathbf{q}}(p))} \tag{10}$$

$$A(\overline{\mathbf{q}}(p))=\overline{\mathbf{q}}^T\mathbf{C}\,\overline{\mathbf{q}}\ ,\qquad B(s(p),\overline{\mathbf{q}}(p))=\overline{\mathbf{q}}^T(2s\mathbf{M}+\alpha\, s^{\alpha-1}p\mathbf{C})\,\overline{\mathbf{q}} \tag{11}$$

The functions $A(\overline{\mathbf{q}}(p))$ and $B(s(p),\overline{\mathbf{q}}(p))$ are expanded in the Taylor's series in the vicinity of $p=0$, i.e.:

$$A(\overline{\mathbf{q}}(p))=A(0)+\left.\frac{\partial A(\overline{\mathbf{q}}(p))}{\partial p}\right|_{p=0},\quad B(s(p),\overline{\mathbf{q}}(p),p)=B(0)+\left.\frac{\partial B(s(p),\overline{\mathbf{q}}(p),p)}{\partial p}\right|_{p=0} \tag{12}$$

where

$$A(0)=\mathbf{a}^T\mathbf{C}\,\mathbf{a}\ ,\qquad B(0)=2\mathrm{i}\omega\,\mathbf{a}^T\mathbf{M}\mathbf{a} \tag{13}$$

Moreover, taking into account that both the eigenvalue s and the eigenvector $\overline{\mathbf{q}}$ depend on p, we can write:

$$\frac{\partial B}{\partial \mathrm{p}} = 2\overline{\mathbf{q}}^T (2s\mathbf{M} + \alpha\, s^{\alpha-1} p\mathbf{C}) \frac{\partial \overline{\mathbf{q}}}{\partial p} + \overline{\mathbf{q}}^T \left[2\mathbf{M} + \alpha\,(\alpha - 1)s^{\alpha-2} p\mathbf{C}\right]\overline{\mathbf{q}}\,\frac{\partial s}{\partial p} + \alpha\, s^{\alpha-1}\overline{\mathbf{q}}^T\mathbf{C}\overline{\mathbf{q}} \quad (14)$$

$$\frac{\partial A}{\partial p} = 2\,\mathbf{q}^T\mathbf{C}\,\frac{\partial \mathbf{q}}{\partial p} \quad (15)$$

The values of the above derivatives at $p = 0$ are:

$$\left.\frac{\partial B}{\partial \mathrm{p}}\right|_{p=0} = 4\mathrm{i}\omega\,\mathbf{a}^T\mathbf{M}\left.\frac{\partial \overline{\mathbf{q}}}{\partial p}\right|_{p=0} + 2\mathbf{a}^T\mathbf{Ma}\left.\frac{\partial s}{\partial p}\right|_{p=0} + \alpha\,(\mathrm{i}\omega)^{\alpha-1}\mathbf{a}^T\mathbf{Ca} \quad (16)$$

$$\left.\frac{\partial A}{\partial p}\right|_{p=0} = 2\,\mathbf{a}^T\mathbf{C}\left.\frac{\partial \mathbf{q}}{\partial p}\right|_{p=0} \quad (17)$$

The sensitivities $\partial\mathbf{q}/\partial p$ and $\partial s/\partial p$, calculated at $p = 0$, can be determined from Eqs (8) and (9). In the vicinity of $p = 0$, these equations take the following form:

$$(\mathbf{K} - \omega^2\mathbf{M})\left.\frac{\partial \overline{\mathbf{q}}}{\partial p}\right|_{p=0} + 2\mathrm{i}\omega\,\mathbf{Ma}\left.\frac{\partial s}{\partial p}\right|_{p=0} = -(\mathrm{i}\omega)^{\alpha}\,\mathbf{C}\,\mathbf{a} \quad (18)$$

$$2\mathrm{i}\omega\,\mathbf{a}^T\mathbf{M}\left.\frac{\partial \overline{\mathbf{q}}}{\partial p}\right|_{p=0} + \mathbf{a}^T\mathbf{Ma}\left.\frac{\partial s}{\partial p}\right|_{p=0} = -\frac{1}{2}\alpha(\mathrm{i}\omega)^{\alpha-1}\mathbf{a}^T\mathbf{Ca} \quad (19)$$

from which the sought quantities could be determined.

Finally, Eq (10) could be rewritten in the form of the following differential equation:

$$\frac{\partial s}{\partial p} = -s^{\alpha}\,\frac{a_0 + a_1 p}{b_0 + b_1 p} \quad (20)$$

where

$$a_0 = \mathbf{a}^T\mathbf{C}\,\mathbf{a}\ , \qquad b_0 = 2\mathrm{i}\omega\,\mathbf{a}^T\mathbf{Ma}\ , \qquad a_1 = 2\,\mathbf{a}^T\mathbf{C}\left.\frac{\partial \mathbf{q}}{\partial p}\right|_{p=0} \quad (21)$$

$$b_1 = 4\mathrm{i}\omega\,\mathbf{a}^T\mathbf{M}\left.\frac{\partial \overline{\mathbf{q}}}{\partial p}\right|_{p=0} + 2\mathbf{a}^T\mathbf{Ma}\left.\frac{\partial s}{\partial p}\right|_{p=0} + \alpha\,(\mathrm{i}\omega)^{\alpha-1}\mathbf{a}^T\mathbf{Ca} \quad (22)$$

It should be noted that, for $p = 0$, the normalization condition (7) is reduced to $2\mathrm{i}\omega\,\mathbf{a}^T\mathbf{Ma} = 1$, which means that $b_0 = 1$. Moreover, from Eq (19)

$$4\mathrm{i}\omega\,\mathbf{a}^T\mathbf{M}\left.\frac{\partial \overline{\mathbf{q}}}{\partial p}\right|_{p=0} + 2\mathbf{a}^T\mathbf{Ma}\left.\frac{\partial s}{\partial p}\right|_{p=0} + \frac{1}{2}\alpha\,(\mathrm{i}\omega)^{\alpha-1}\mathbf{a}^T\mathbf{Ca} = 0 \quad (23)$$

which means that $b_1 = 0$.

Finally, Eq (20) is reduced to

$$\frac{\partial s}{\partial p} = -s^{\alpha}(a_0 + a_1 p) = f(s, p) \tag{24}$$

and only the constant a_1 depends on α .

The solution to Eq (24) must fulfill the following initial condition: for $p = 0$ $s(0) = \mathrm{i}\omega$ or $s(0) = -\mathrm{i}\omega$, depending on which complex conjugate solution is sought.

Before describing the method for solving Eq (24), the special case $\alpha = 1$ will be discussed. It means that dampers are described by classic rheological models and Eq (24) takes the following form:

$$\frac{\partial s}{\partial p} = -s\,(a_0 + a_1 p) \tag{25}$$

Solution to Eq (25) fulfilling the described above initial conditions is given by

$$s(p) = \pm\mathrm{i}\omega \exp\,(a_0 p + \tfrac{1}{2} a_1 p^2) \tag{26}$$

It means that, for $p = 1$, the following approximate solution to the eigenvalue of the nonlinear eigenvalue problem (4) is obtained:

$$\hat{s} = \pm\mathrm{i}\omega \exp\,(a_0 + \tfrac{1}{2} a_1) \tag{27}$$

The above result is identical with the one obtained in [4].

An implicit version of the Euler method is used to solve Eq (24) numerically. First of all, the increment of p is chosen and denoted by h. Moreover, a set of points are chosen on the p axis in such a way that $p_{n+1} = p_n + h$ and the notation $s(p_n) = s_n$ is used. According to the Euler method

$$s_{n+1} = s_n + [f(s_n, p_n) + f(s_{n+1}, p_{n+1})]\,h/2 \tag{28}$$

and for $n = 0$, $s(0) = s_0 = \pm\mathrm{i}\omega$.

The simple iteration method is adopted for solving the nonlinear algebraic equation (28) with respect to s_{n+1}. The initial approximation of s_{n+1} is calculated from the formula:

$$s_{n+1}^{(0)} = s_n + hf(s_n, p_n) \tag{29}$$

and the (i+1)-th approximation of s_{n+1} is given by

$$s_{n+1}^{(i+1)} = s_n + [f(s_n, p_n) + f(s_{n+1}^{(i)}, p_{n+1})]\,h/2 \tag{30}$$

where the superscript denotes the number of iteration.

The iteration is continued until the following inequality is fulfilled:

$$\left|\mathrm{Re}(s_{n+1}^{(i+1)} - s_{n+1}^{(i+1)})\right| \le \varepsilon \left|\mathrm{Re}(s_{n+1}^{(i+1)})\right| , \qquad \left|\mathrm{Im}(s_{n+1}^{(i+1)} - s_{n+1}^{(i+1)})\right| \le \varepsilon \left|\mathrm{Im}(s_{n+1}^{(i+1)})\right| \tag{31}$$

where ε is the assumed accuracy of calculation.

Having the eigenvalue $s = \mu + \mathrm{i}\eta$, the natural frequency ω and the non-dimensional damping ratio γ is determined from

$$\omega^2 = \mu^2 + \eta^2 , \qquad \gamma = -\mu/\omega \tag{32}$$

The approximation of eigenvector $\bar{\mathbf{q}}$ is given by

$$\bar{\mathbf{q}} = \mathbf{a} + \left.\frac{\partial \bar{\mathbf{q}}}{\partial p}\right|_{p=0} \tag{33}$$

4. Representative results

Results for a four-storey shear frame with two dampers located at the first and fourth storeys will be presented. The fractional Kelvin model is used for describing the dampers. The following data are used for describing the frame: (i) the storeys' stiffness are: $k_1 = k_2 = 26.0 \cdot 10^6$ [N/m], $k_3 = k_4 = 20.0 \cdot 10^6$ [N/m]; (ii) the storeys' masses are: $m_1 = m_2 = m_3 = m_4 = 34.0 \cdot 10^3$ [kg]. The first-floor damper's parameters are: $\alpha = 0.8$, $k_{0,1} = 10.0 \cdot 10^6$ [N/m], $c_{0,1} = 0.4 \cdot 10^6$ [Ns$^\alpha$/m] and the damper's parameters for the fourth floor are: $\alpha = 0.8$, $k_{0,2} = 6.0 \cdot 10^6$ [N/m], $c_{0,2} = 0.2 \cdot 10^6$ [Ns$^\alpha$/m].

The system matrices are: $\mathbf{M} = diag[m_1,\ m_2,\ m_3,\ m_4]$,

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 + k_{0,1} & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_{0,2} & -k_4 - k_{0,2} \\ 0 & 0 & -k_4 - k_{0,2} & k_4 + k_{0,2} \end{bmatrix} \tag{34}$$

$$\mathbf{C} = \begin{bmatrix} c_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_{0,2} & -c_{0,2} \\ 0 & 0 & -c_{0,2} & c_{0,2} \end{bmatrix} \tag{35}$$

The natural frequencies of frame without dampers are in the first column of Table 1, whereas in the next column, there are the natural frequencies resulting from Eq (6) when the stiffness matrix is the sum of the stiffness of frame and dampers.

Table 1. Natural frequencies of frame

Frame without dampers	From Eq (6)	Difference
$\omega_1 = 9.27367$ rad/s	$\omega_1 = 9.90121$ rad/s	6.77%
$\omega_2 = 25.35547$ rad/s	$\omega_2 = 27.9594$ rad/s	10.27%
$\omega_3 = 39.20204$ rad/s	$\omega_3 = 43.15206$ rad/s	10.07%
$\omega_4 = 48.9857$ rad/s	$\omega_4 = 50.51930$ rad/s	3.13%

Results of calculation are presented in Table (2). The exact vales of eigenvalues are obtained by means of the continuation method described in [1]. The second column collect results obtained by means of Eq (24). Very similar results are obtained and differences are not greater than 0.2 %. Moreover, in Table 3, the exact and approximate natural frequencies and non-dimensional damping ratios are compared. It is evident that the approximate results have an very good accuracy.

Although in Table 4, the eigenvalues of frame with dampers are stated, now, the dampers are described with the help of a classic Kelvin model ($\alpha = 1.0$, and other damping data are as stated previously). In the second column, the eigenvalues calculated from Eq (27) are presented. Moreover, the exact values of natural frequencies and non-dimensional damping ratios are shown. Exact eigenvalues are obtained using the classical approach given in [6].

A comparison of the eigenvalues obtained from the formula derived by Lazaro in [4] with the ones resulting as the solution to the differential equation (24) is presented in Table 5.

Table 2. Eigenvalues for a frame with the fractional Kelvin dampers ($\alpha = 0.8$)

Eigenvalues (exact results)	Eigenvalues – Euler Eq (24)	Differences
$s_{1,5} = -0.112501 \pm \mathrm{i}\ 9.94280$	$s_{1,5} = -0.11262 \pm \mathrm{i}\, 9.94295$	$0.11\% + \mathrm{i}\, 0.00\%$
$s_{2,6} = -1.11279 \pm \mathrm{i}\ 28.3731$	$s_{2,6} = -1.11240 \pm \mathrm{i}\, 28.3720$	$0.03\% + \mathrm{i}\, 0.00\%$
$s_{3,7} = -2.61179 \pm \mathrm{i}\, 43.9430$	$s_{3,7} = -2.61042 \pm \mathrm{i}\, 43.9405$	$0.05\% + \mathrm{i}\, 0.00\%$
$s_{4,8} = -1.58018 \pm \mathrm{i}\, 50.9182$	$s_{4,8} = -1.57899 \pm \mathrm{i}\, 50.9174$	$0.08\% + \mathrm{i}\, 0.00\%$

Table 3. Natural frequencies and non-dimensional damping ratios – comparison of exact and approximate results for a frame with the fractional Kelvin dampers ($\alpha = 0.8$)

Frequency (exact results)	Damping ratio (exact results)	Frequency (approximate results)	Approximate damping ratio
$\omega_1 = 9.94344$ rad/s	$\gamma_1 = 0.01131$	$\omega_1 = 9.94359$ rad/s	$\gamma_1 = 0{,}01133$
$\omega_2 = 28.3949$ rad/s	$\gamma_2 = 0.03919$	$\omega_2 = 28.3938$ rad/s	$\gamma_2 = 0.03918$
$\omega_3 = 44.0205$ rad/s	$\gamma_3 = 0.05933$	$\omega_3 = 44.0118$ rad/s	$\gamma_3 = 0.05930$
$\omega_4 = 50.9427$ rad/s	$\gamma_4 = 0.03102$	$\omega_4 = 50.9418$ rad/s	$\gamma_4 = 0.03100$

Table 4. Eigenvalues for frame with classic Kelvin dampers ($\alpha = 1.0$)

Eigenvalues (exact results)	Eigenvalues - Eq (27)	Frequency [rad/s]	Damping ratio
$s_{1,5} = -0.19179 \pm \mathrm{i}\, 9.9133$	$s_{1,5} = -0.19196 \pm \mathrm{i}\, 9.88545$	$\omega_1 = 9.91516$	$\gamma_1 = 0.019343$
$s_{2,6} = -2.3712 \pm \mathrm{i}\, 28.114$	$s_{2,6} = -2.31103 \pm \mathrm{i}\, 27.6169$	$\omega_2 = 28.2138$	$\gamma_2 = 0.084044$
$s_{3,7} = -5.7509 \pm \mathrm{i}\, 42.729$	$s_{3,7} = -5.73843 \pm \mathrm{i}\, 42.8072$	$\omega_3 = 43.1142$	$\gamma_3 = 0.133387$
$s_{4,8} = -3.4508 \pm \mathrm{i}\, 49.919$	$s_{4,8} = -3.45254 \pm \mathrm{i}\, 49.9294$	$\omega_4 = 50.0381$	$\gamma_4 = 0.068963$

5. Concluding remarks

The proposed method enables determination of the dynamic properties of structures with VE dampers in a simple way. The dampers' behavior is described with the help of fractional derivatives. A partial solution to the classic eigenvalue problem is necessary in the proposed method. Only one eigenvector and the corresponding eigenvalue of problem (6) are necessary to determine the conjugated eigenvalue and eigenvector for the structure with VE dampers. The results of an extensive calculation, which is not presented in this paper due to the limitation of space, indicate that the accuracy of the method is good for a range of damper's parameters used in practice.

Table 5. Eigenvalues for frame with classic Kelvin dampers ($\alpha = 1.0$) – comparison of the results obtained from Eq (24) (the Euler method) and Eq (27)

Eigenvalues – Euler Eq (24)	Eigenvalues - Eq (27)	Differences
$s_{1,5} = -0.19250 \pm \mathrm{i}\,9.91326$	$s_{1,5} = -0.19196 \pm \mathrm{i}\,9.88545$	$0.28\% + \mathrm{i}\,0.28\%$
$s_{2,6} == -2.35233 \pm \mathrm{i}\,28.1094$	$s_{2,6} = -2.31103 \pm \mathrm{i}\,27.6169$	$1.79\% + \mathrm{i}\,1.78\mathrm{i}\,\%$
$s_{3,7} = -5.72822 \pm \mathrm{i}\,42.7318$	$s_{3,7} = -5.73843 \pm \mathrm{i}\,42.8072$	$0.18\% + \mathrm{i}\,0.18\%$
$s_{4,8} = -5.55776 \pm \mathrm{i}\,50.1042$	$s_{4,8} = -5.58168 \pm \mathrm{i}\,50.3188$	$0.43\% + \mathrm{i}\,0.43\%$

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