# Influence of Substructure Properties on Natural Vibrations of Periodic Euler-Bernoulli Beams

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**Abstract**

In this paper there are considered vibrations of Euler-Bernoulli beams with geometrical and material properties periodically varying along the axis. The basic exact equations with highly oscillating periodic coefficients are replaced by the system of averaged equations with constant coefficients. The new model is based on the tolerance modelling technique, which describes macro-dynamics of the beam including the effect of the microstructure size. The purpose of this paper is to present an approximately equivalent model, which describe vibrations of periodic beams taking into account length of the periodicity cell.

***Keywords*:** periodic beams, Euler-Bernoulli beams

## 1. Introduction

This paper is related to certain problems, which are met in the analysis of periodic beams. Dynamics of such beams is described by differential equations with non-continuous highly oscillating periodic coefficients. Therefore, various approximate models, introducing effective beam properties are proposed. Amongst them, can be mentioned those, based on the asymptotic homogenization, cf. [1, 2, 7]. However, in many technical problems, number of cells is finite. Thus, neglecting the microstructure size may lead to erroneous results, especially in the range of high frequencies.

In order to include the effect of microstructure size, the tolerance modelling technique is introduced (cf. the book edited by Cz. Woźniak, Michalak and Jędrysiak [10]). The preceding method is very general and convenient for modelling problems, described by differential equations with highly oscillating coefficients, e.g. modelling of dynamic behaviour of microstructured thin functionally graded plates [6] and dynamic problems for plates with a periodic structure [8]. In contrary to the exact solutions, the obtained relations have constant coefficients, some of which explicitly depend on the microstructure size.

Wave propagation and linear vibrations in periodic beams are revised in many research papers. For a periodic Euler-Bernoulli beam it is considered in [3] and [9].

Frequency band gaps were analyzed by the differential quadrature method in [11]. The transfer matrix method was applied in [12] in analysis of flexural wave propagation in the beam on elastic foundation. In [4] a wide literature study on composite beam vibration can be found. In order to determine a homogenized model of a composite beam with small periodicity the two-scale asymptotic expansion method is used in [5].

In this paper the tolerance model of Euler-Bernoulli beam with geometrical and material properties periodically varying along the axis is presented and discussed. The tolerance averaging model is applied to investigate free vibration frequencies for an Euler-Bernoulli beam. Obtained results are compared with finite element method.

## 2. Formulation of the problem

Let *Oxyz* be an orthogonal Cartesian coordinate system, the *Ox* axis coincides with the axis of the beam. It is assumed that considered elastic periodically inhomogeneous Euler-Bernoulli beam consists of many small repetitive elements called periodicity cells. It is also assumed that every such element can be treated as an Euler-Bernoulli beam. Hence, it is defined the region $\Omega \equiv [0, L]$, where $L$ is the beam length. The considered cells are defined as $\Delta \equiv [-l/2, l/2]$, where $l \ll L$ is the dimension of the cell, called microstructure parameter. It is assumed that the beam possesses principal planes and that the vibration takes place in one of the principal planes. Let $w = w(x,t)$ denote the small deflection of the neutral axis of the beam from its initial, straight configuration. The following notation is introduced: $\partial^k = \partial^k / \partial x^k$ is the $k$-th derivative with respect to the $x$ coordinate and overdot stands for the derivative with respect to time. For small deflections of the beam strain and kinetic energy are:

$$U = \frac{1}{2}\int_0^L EJ\partial^2 w \partial^2 w dx, \quad K = \frac{1}{2}\int_0^L \mu \dot{w}\dot{w} dx, \tag{1}$$

where $E = E(x)$, $J = J(x)$, $\mu = \mu(x)$ are the Young's modulus of the beam material, the cross-sectional moment of inertia, the mass per unit length of the beam, respectively. Since only free vibrations are considered, the potential energy of the external load is assumed to be zero.

The equation of motion is derived from Hamilton's principle:

$$\delta \int_{t_0}^{t_1} \mathcal{L}\, dt = \delta \int_{t_0}^{t_1} (U - K)\, dt = 0. \tag{2}$$

The Lagrange function for the problem can be written as:

$$\mathcal{L} = \frac{1}{2} EJ\partial^2 w \partial^2 w - \frac{1}{2}\mu \dot{w}\dot{w}. \tag{3}$$

Following the usual procedure of the calculus of variations, the Euler equation of motion is obtained:

$$\partial^2\left(EJ\partial^2 w\right) + \mu \ddot{w} = 0. \tag{4}$$

The coefficients $E$, $J$, $\mu$, are in considered cases highly oscillating, non-continuous functions of the $x$-coordinate.

## 3. The tolerance averaging approach – introductory concepts and basic assumptions

The main concept of tolerance averaging approach is the tolerance reflexive relation. Amongst the fundamental ideas of the technique the most remarkable are certain classes of functions such as the tolerance-periodic (*TP*), slowly-varying (*SV*), highly oscillating (*HO*) and fluctuation shape (*FS*) function.

A cell at $x \in \Omega_\Delta$ is denoted by $\Delta(x) = x + \Delta,\ \Omega_\Delta = \{x \in \Omega : \Delta(x) \subset \Omega\}$. The averaging operator for an arbitrary integrable function *f* is defined by

$$\langle f \rangle(x) = \frac{1}{l} \int_{\Delta(x)} f(y)dy, \quad x \in \Omega_\Delta, \quad y \in \Delta(x). \tag{5}$$

The basic assumption of micro-macro decomposition plays imperative role in tolerance modelling technique. The unknown transverse deflection can be decomposed into their averaged and fluctuating part:

$$\begin{aligned} &w(x,t) = W(x,t) + h^A(x)V^A(x,t), \quad A = 1,\ldots,N, \\ &W(\cdot), V^A(\cdot) \in SV_d^2(\Omega,\Delta), \quad h^A(\cdot) \in FS_d^2(\Omega,\Delta), \end{aligned} \tag{6}$$

where $W(\cdot)$ (macrodeflection) and $V^A(\cdot)$ (fluctuation amplitudes of the deflection) functions are the basic unknowns; $h^A$ is the known fluctuation shape function. The tolerance parameter, associated with the tolerance relation, is denoted by $d$, $0<d<<1$. It is assumed that the unknown functions are slowly-varying (*SV*) up to the second derivative, which is denoted by the top index.

The highly oscillating fluctuation shape functions (*FSFs*) $h^A$, proposed *a priori* for every considered problem, are assumed to describe the unknown fields oscillations caused by the structure inhomogeneity. What is more, *FSFs* have to ensure the *l*-periodicity constraint and provide the conditions below:

$$\left\langle \mu h^A \right\rangle = 0, \quad \left\langle \mu h^A h^B \right\rangle = 0 \ \text{for A} \neq \text{B}; \quad \partial^m h^A \in O\left(l^{2-m}\right), \quad A,B = 1,\ldots,N. \tag{7}$$

## 4. Governing equations of the model

### 4.1. Tolerance model equations

In the first place, the micro-macro decomposition (6) of Lagrangian (3) is performed. Next, averaging over an arbitrary periodicity cell is performed (5), applying the aforementioned approximations (7).

The variation of averaged functional has the specified form:

$$\delta \int_{t_0}^{t_1}\int_0^L \langle \mathcal{L}_h \rangle dxdt = \int_{t_0}^{t_1}\int_0^L \delta \langle \mathcal{L}_h \rangle dxdt. \tag{8}$$

Therefore, after expanding we obtain:

$$\begin{aligned} \delta \int_{t_0}^{t_1}\int_0^L \langle \mathcal{L}_h \rangle dxdt = \delta \tfrac{1}{2} \int_{t_0}^{t_1}\int_0^L \Big[ &\langle EJ \rangle \partial^2 W \partial^2 W + 2\left\langle EJ\partial^2 h^A \right\rangle \partial^2 W V^A + \\ + \left\langle EJ\partial^2 h^A \partial^2 h^B \right\rangle V^A V^B - \langle \mu \rangle \dot{W}\dot{W} - 2\left\langle \mu h^A \right\rangle &\dot{W}\dot{V}^A - \left\langle \mu h^A h^B \right\rangle \dot{V}^A \dot{V}^B \Big] dxdt. \end{aligned} \tag{9}$$

From the principle of stationary action, applied to the averaged Lagrangian, the averaged Euler-Lagrange equations are obtained:

$$
\begin{aligned}
&\langle EJ\rangle\partial^4 W + \left\langle EJ\partial^2 h^A\right\rangle\partial^2 V^A + \langle\mu\rangle\ddot{W} + \underline{\left\langle \mu h^A\right\rangle}\ddot{V}^A = 0,\\
&\left\langle EJ\partial^2 h^A\right\rangle\partial^2 W + \left\langle EJ\partial^2 h^A\partial^2 h^B\right\rangle V^B + \underline{\left\langle \mu h^A\right\rangle}\ddot{W} + \underline{\left\langle \mu h^A h^B\right\rangle}\ddot{V}^B = 0.
\end{aligned} \tag{10}
$$

In contrast to the exact formulation (4), obtained system of 1+$N$ differential equations for the macrodisplacement $W(\cdot)$ and fluctuation amplitudes of deflection $V^A(\cdot)$ has constant coefficients. Underlined coefficients depend on the microstructure parameter $l$. In order to present (10) in more convenient form, let us denote coefficients by:

$$
\begin{Bmatrix} D \\ D^A \\ D^{AB} \end{Bmatrix} \equiv \begin{Bmatrix} \langle EJ\rangle \\ \left\langle EJ\partial^2 h^A\right\rangle \\ \left\langle EJ\partial^2 h^A\partial^2 h^B\right\rangle \end{Bmatrix}, \quad \begin{Bmatrix} M \\ M^A \\ M^{AB} \end{Bmatrix} \equiv \begin{Bmatrix} \langle\mu\rangle \\ \left\langle \mu h^A\right\rangle \\ \left\langle \mu h^A h^B\right\rangle \end{Bmatrix}. \tag{11}
$$

After taking into account (11) we get:

$$
\begin{aligned}
&D\partial^4 W + D^A\partial^2 V^A + M\ddot{W} = 0,\\
&D^A\partial^2 W + D^{AB}V^B + \underline{M^{AB}}\ddot{V}^B = 0,
\end{aligned} \tag{12}
$$

where $M^{AB}$ depends on microstructure size.

### 4.2. Asymptotic model equations

The asymptotic tolerance model is obtained by neglecting coefficients dependent on microstructure size $l$. If matrix $D^{AB}$ is nonsingular, then there exists an inverse matrix $(D^{AB})^{-1}$.Thus, let us denote the effective stiffness of the beam by:

$$
D_0 = D - D^A\left(D^{AB}\right)^{-1}D^B. \tag{13}
$$

Therefore the asymptotic model equations become

$$
\begin{aligned}
&D_0\partial^4 W + M\ddot{W} = 0,\\
&V^A = -\left(D^{BA}\right)^{-1}D^B\partial^2 W.
\end{aligned} \tag{14}
$$

## 5. Natural frequencies

We can transform system of PDEs (12) into system of ODEs using separation of variables. Let us expand macrodeflection and fluctuation amplitudes of the deflection into series of eigenfunctions of a simply supported beam:

$$
\begin{Bmatrix} W(x,t) \\ V^A(x,t) \end{Bmatrix} = \sum_{m=1}^{\infty}\begin{Bmatrix} w_m(t) \\ v_m^A(t) \end{Bmatrix}\sin\xi_m x, \quad \xi_m = \frac{m\pi}{L}. \tag{15}
$$

Substituting (16) into (13) and limiting the analysis for one $FSF$ we obtain:

$$
\begin{aligned}
&\xi_m^4 D w_m - \xi_m^2 D^1 v_m^1 + M\ddot{w}_m = 0,\\
&-\xi_m^2 D^1 w_m + D^{11}v_m^1 + \underline{M^{11}}\ddot{v}_m^1 = 0.
\end{aligned} \tag{16}
$$

There can be assumed the following solutions:

$$\begin{Bmatrix} w_m \\ v_m^A \end{Bmatrix} = \begin{Bmatrix} A_m^W \\ A_m^{V^1} \end{Bmatrix} \cos \omega t. \tag{17}$$

In order to find free vibrations frequencies, we introduce subsequent symbols:

$$\tilde{a}_m \equiv \frac{\xi_m^4 D}{M}, \quad \tilde{b}_m \equiv -\frac{\xi_m^2 D^1}{M}, \quad \tilde{c}_m \equiv -\frac{\xi_m^2 D^1}{M^{11}}, \quad \tilde{d} \equiv \frac{D^{11}}{M^{11}}. \tag{18}$$

System of equations (17) is in fact, an eigenvalue problem:

$$\begin{bmatrix} \tilde{a}_m - \omega^2 & \tilde{b}_m \\ \tilde{c}_m & \tilde{d} - \omega^2 \end{bmatrix} \begin{Bmatrix} w_m \\ v_m^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{19}$$

We can obtain expressions for high and low natural frequencies by finding the roots of characteristic polynomial:

$$\left(\omega_m^{-,+}\right)^2 = \frac{1}{2}\left[\tilde{a}_m + \tilde{d} \mp \sqrt{\left(\tilde{a}_m - \tilde{d}\right)^2 + 4\tilde{b}_m \tilde{c}_m}\,\right]. \tag{20}$$

## 6. Examples of applications

### 6.1. Introduction

The object under consideration is a hinged-hinged beam, which fragment is shown in Fig. 1. The beam's cross section, moment of inertia, Young's modulus and mass per unit length are periodically varying along the axis. It is assumed that cross section of the beam is rectangular. Considered periodicity cell, presented in Fig. 2, has symmetrical shape. Length of its segments depends on an $\alpha$ parameter.

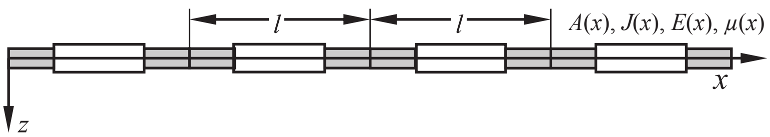


Figure 1. Fragment of considered periodic Euler-Bernoulli beam

The fluctuation shape functions represent the oscillations of displacements within the periodicity cell. For a purposes of this paper there were used approximate $l$-periodic trigonometric functions: $h^1(y) = l^2[\cos(2\pi y/l)+c]$.

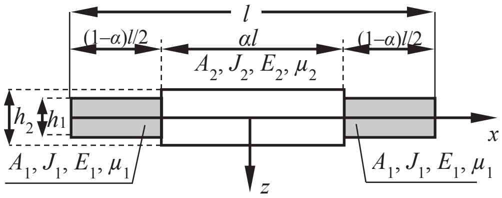


Figure 2. Periodicity cell

### 6.2. Results and discussion

The free vibrations of a slender periodic beam depending on the $\alpha$ parameter are considered. The calculations are carried out for two cases:

1. Constant geometrical properties and periodically varying values of mass density and Young's modulus.
2. Constant material properties and periodically varying height of beam's cross section.

In both cases it is assumed that considered beam has following properties: length $L = 1.0$ m; periodicity cell's length $l = 1/10L = 10$ cm.

For the first problem it is assumed that Young's modulus $E_1 = E = 210$ GPa; $E_2 = [0.25, 0.50, 0.75]E$; mass density $\rho_1 = \rho = 7860$ kg/m$^3$; $\rho_2 = [0.25, 0.50, 0.75]\rho$; cross section width and height: $b = 2$ cm, $h = 2$ cm. The results are shown in Fig. 3 and Fig. 4. It is evident that TAT has the best agreement with FEM for less disproportion of material parameters. For $E_2 = 0.75E$ and $E_2 = 0.50E$ the solutions are almost equal.

In the second case we declare following properties: $E = 210$ GPa, $\rho = 7860$ kg/m$^3$; $b = 2$ cm; $h_1 = h = 2$ cm, $h_2 = [0.50, 0.70, 0.90]h$. Figure 4(a) shows the results for this particular case. It is evident that difference in stiffness of the beam's segments is noticeably high. Similarly, as it was earlier, the proposed method delivered the best results for less disproportion of given properties. It is evident that tolerance model in cases with high disproportion is stiffer that FEM. The maximum value of obtained frequencies from both tolerance averaging method and FEM is denoted by $\omega_{\max}$. What is more, TAT gives the opportunity to analyse higher natural frequencies, as it is shown in Fig. 4(b). Study based on the finite element method does not provide such a possibility.

The assumed tolerance averaging model has 2 degrees of freedom and approximate fluctuation shape functions. It is worth noting, that comparative finite element model has 30 elements and 60 degrees of freedom.

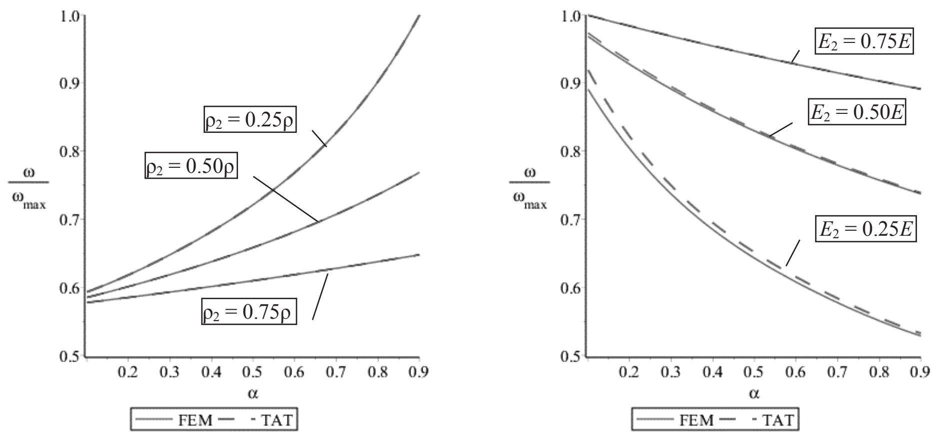


Figure 3. First natural frequencies for various values of mass density and Young's modulus

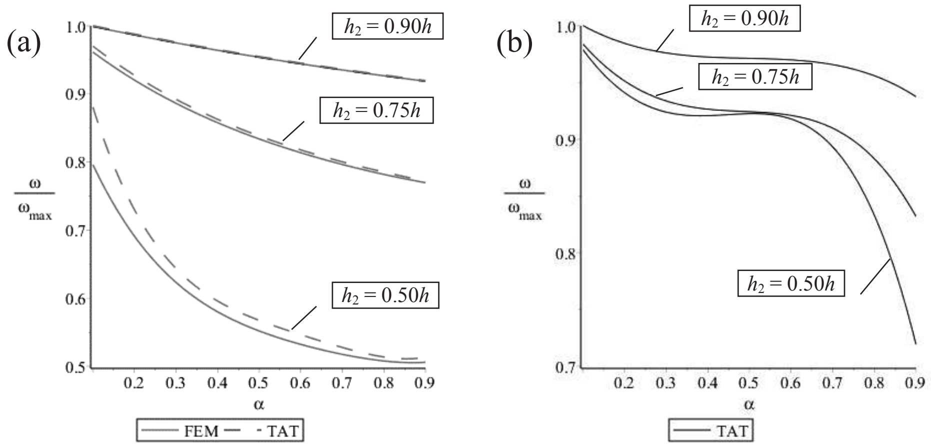


Figure 4. First lower (a) and higher (b) natural frequencies for various values of cross section height

## 7. Conclusions

The free vibrations of Euler-Bernoulli beams, with geometrical and material properties periodically varying along the axis have been considered. The model equations are obtained by implementing the tolerance averaging technique. Derived differential equations have constant coefficients. The main advantage of this approach is that it includes the effect of the period lengths on the overall behaviour of these beams. Despite the use of the approximate fluctuation shape functions the results are consistent with finite element method.

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