Evaluation of the Transmission and the Scattering Matrix Applicability to the Mufflers Analysis

Anna SNAKOWSKA
AGH University of Science and Technology, Faculty of Mechanical Engineering and Robotics, Krakow, Poland, anna.snakowska@agh.edu.pl

Łukasz GORAZD
AGH University of Science and Technology, Faculty of Mechanical Engineering and Robotics, Krakow, Poland, lukasz.gorazd@agh.edu.pl

Jerzy JURKIEWICZ
AGH University of Science and Technology, Krakow, Poland, jerzy.jurkiewicz@agh.edu.pl

Abstract

The analysis of the acoustical systems can be carried out based on a number of different formalisms, of which applied frequently are the transfer matrix formalism, in which the chosen state variables are the sound pressure p and the sound velocity v, and the scattering matrix formalism adopting the sound pressures p^+ and p^- and the sound velocities v^+ and v^- of waves propagating through an element in both directions. Even though, they are mathematically equivalent, i.e. there exists the unequivocal transformation from one to another there are some advantages and disadvantages in applying one or the other to analyse mufflers or other acoustic system, especially when the propagation of a multimode wave is assumed and numerical calculations are indispensable. In the paper the transformation between the formalisms has been derived and applied to analyse the phenomena at a junction between the tail pipe and the chamber and also in mufflers composed of one or two chambers. The more flexible for numerical calculations seems the scattering matrix formalism, especially when the number of propagating modes differs on both sides of a junction. On the other hand the transmission matrix formalism is suitable for analysing systems constituting a cascade. The sources of the advantages and disadvantages of both formalisms are explained. The results obtained can be helpful in the effective design of silencers with specific properties.

Keywords: transmission matrix, scattering matrix, multimode wave, silencers and mufflers

1. Introduction

The examination of the properties of acoustic systems, in particular silencers, can be carried out using the transmission, impedance, admittance or the scattering matrix. The use of one of these formalisms depends on the choice of the state variables. In each of these formalisms, a relationship is sought between the state variables. In the study of muffler attenuation properties, these are the values of state variables on selected cross-sections of inlet and outlet pipes. In the case of analysing the wave propagation through all the elements of the muffler in the low frequencies approximation, i.e. assuming only the plane wave propagation, these matrices have the dimension of 2 x 2. However, the analysis carried out in this way is correct only if the Helmholtz number for each of the muffler elements does not exceed the cut-off frequency of the first waveguide mode.

In the case of silencers with cylindrical symmetry, it is the value of ka = 1.84 for axial asymmetric excitation and ka = 3.83 for symmetric excitation. However, even if these conditions are fulfilled, the analysis will be subject to some errors resulting from neglecting the effects of the near field on the discontinuities of the boundary condition, *i.e.* for example on the junction of the inlet/outlet pipe with the chamber.

Formalities previously mentioned are mathematically equivalent, that is there is unequivocal transformation of one matrix to another. However, they are not equivalently convenient to adopt, especially if the muffler testing goes beyond the plane wave approximation, accounting the propagation of the higher waveguide modes. For cylindrical symmetry, this will be the so-called Bessel modes.

The article analyses two formalisms in terms of their advantages and disadvantages -- the formalism of the $\boldsymbol{T}$ transmission matrix and the $\boldsymbol{S}$ scattering matrix. The analysis was carried out for silencers consisting of one or more chambers with different radii. For a given frequency, the number of propagating modes (cut-on modes) depends on the radius of an element, so it can change from one element to another. This fact, as will be shown, significantly impedes the analysis by means of the transmission matrix $\boldsymbol{T}$. On the other hand, the $\boldsymbol{T}$ matrix is particularly convenient to describe the cascade system of which it is a product of individual elements' matrices [1]. There scattering matrix $\boldsymbol{S}$ do not have this convenient property, but in turn it can be adopted without problems for a variable number of modes propagating in individual elements of the silencer [2, 3].

The silencers have been subject of numerous research and publications. At first they have been examined within the plane wave approximation [1, 4-6], however of increasingly complex geometry [5, 6] and with application of additional sound attenuating materials such as linings or perforates surfaces [7, 8]. Constructing mufflers of large radii or adapted to suppress high frequencies caused the necessity of analyzing the propagation of modes other than the plane wave [2, 3, 9-11]. Frequently, the multimode wave has been assumed in chambers and the incident plane wave in the inlet pipe [9, 11]. To adjust the acoustic pressure and the velocity at junctions between subsequent elements the mode matching method (MMM) has been applied [9-12].

2. Transformation of the transmission matrix into the scattering matrix

Let us consider a simple muffler composed of two tail pipes (inlet and outlet) of the same radius a connected to the expansion chamber of the radius b in which a multimode wave propagates (Fig. 1).

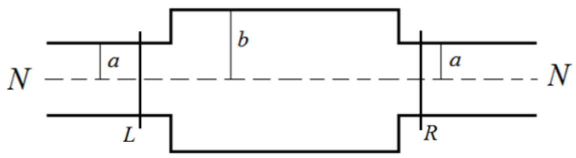

Figure 1. Schematic representation of a simple muffler

The relation between the modal pressures and modal velocities on cross sections depicted as L (left) and R (right) are given by the transfer matrix.

$$\begin{bmatrix} \boldsymbol{P^R} \\ \boldsymbol{V^R} \end{bmatrix} = \boldsymbol{T} \times \begin{bmatrix} \boldsymbol{P^L} \\ \boldsymbol{V^L} \end{bmatrix} \tag{1}$$

where

$$\begin{bmatrix} \boldsymbol{P^R} \\ \boldsymbol{V^R} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_1^R} \\ \vdots \\ \boldsymbol{P_N^R} \\ \boldsymbol{V_1^R} \\ \vdots \\ \boldsymbol{V_N^R} \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{P^L} \\ \boldsymbol{V^L} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_1^L} \\ \vdots \\ \boldsymbol{P_N^L} \\ \boldsymbol{V_1^L} \\ \vdots \\ \boldsymbol{V_N^L} \end{bmatrix} \tag{2}$$

The elements of the muffler form a cascade, and so the transfer matrix of the system is a product of transfer matrices representing subsequent elements, which are: a straight duct of a given length, a junction from a tail pipe to an expansion chamber, then throught the expansion chamber to the junction on the right and finally along the exhaust pipe. The mathematical formula of this cascade transfer matrix is

$$\boldsymbol{T} = \boldsymbol{T}_1 \times \boldsymbol{T}_{M1} \times \boldsymbol{T}_2 \times \boldsymbol{T}_{M2} \times \boldsymbol{T}_3 \tag{3}$$

where $\boldsymbol{T}_1 = \boldsymbol{T}_3$ and $\boldsymbol{T}_{M1} = \boldsymbol{T}_{M3}^{-1}$.The transfer matrices $\boldsymbol{T}_1, \boldsymbol{T}_2$ and $\boldsymbol{T}_3$ express the change of phase of the wave travelling across a straight duct that is

$$\boldsymbol{T}_i = \begin{bmatrix} \left[\mathrm{diag}\left(\cos\left(k_{z,n}^i l^i\right)\right)\right] & \left[\mathrm{diag}\left({}^{i}/_{Y_n^a} \cdot \sin\left(k_{z,n}^i l^i\right)\right)\right] \\ \left[\mathrm{diag}\left(\mathrm{i}Y_n^i \cdot \sin\left(k_{z,n}^i l^i\right)\right)\right] & \left[\mathrm{diag}\left(\cos\left(k_{z,n}^i l^i\right)\right)\right] \end{bmatrix} \tag{4}$$

where $k_{z,n}^i$ is the axial wave number of the *n*-th mode in *i*-th straight element of length l^i and the transmission matrix of a junction $\boldsymbol{T}_M$ is

$$\boldsymbol{T}_M = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix} \tag{5}$$

where the elements of the **F** and **G** matrices have been calculated applying the mode mat-ching method [3] at the junction and are equal

$$F_{p,m} = \frac{1}{S} \iint\limits_{S_B} \Psi_m^A(\varrho,\varphi)\Psi_p^{B*}(\varrho,\varphi)\varrho d\varrho d\varphi\,, \quad m = 1,2,3\,... \tag{6}$$

$$G_{m,p} = \frac{1}{S} \iint\limits_{S_B} \Psi_p^B(\varrho,\varphi)\Psi_m^{A*}(\varrho,\varphi)\varrho d\varrho d\varphi, \quad m = 1,2,3\,... \tag{7}$$

where $\Psi_m^A(\varrho,\varphi)$ and $\Psi_p^B(\varrho,\varphi)$ are the mode shape functions of the tail pipes and the chamber, respectively.

The scattering matrix describes the relation between the ingoing and outgoing modal pressures of the element under consideration („black box").

$$\begin{bmatrix}\boldsymbol{P^{L-}}\\ \boldsymbol{P^{R+}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{S^{11}} & \boldsymbol{S^{12}}\\ \boldsymbol{S^{21}} & \boldsymbol{S^{22}}\end{bmatrix} \times \begin{bmatrix}\boldsymbol{P^{L+}}\\ \boldsymbol{P^{R-}}\end{bmatrix} \tag{8}$$

where $\boldsymbol{P^{L+}}$ and $\boldsymbol{P^{R-}}$ represent the one column matrices of the in-going waves modal pressures on the selected cross sections of the muffler (*cf.* Fig. 1), that is

$$\begin{bmatrix}\boldsymbol{P^{L-}}\\ \boldsymbol{P^{R+}}\end{bmatrix} = \begin{bmatrix}P_1^{L-}\\ \vdots\\ P_N^{L-}\\ P_1^{R+}\\ \vdots\\ P_N^{R+}\end{bmatrix} \qquad \begin{bmatrix}\boldsymbol{P^{L+}}\\ \boldsymbol{P^{R-}}\end{bmatrix} = \begin{bmatrix}P_1^{L+}\\ \vdots\\ P_N^{L+}\\ P_1^{R-}\\ \vdots\\ P_N^{R-}\end{bmatrix} \tag{9}$$

On the left side the ingoing waves propagate in the positive direction of the muffler axis, while on the right side cross section in the opposite direction and that determines the signs „+" and „ – " .

Derivation of the transfer matrix allows calculating the scattering matrix by means of the linear transformation. The procedure of transforming the transfer matrix $\boldsymbol{T}$ into the scattering matrix $\boldsymbol{S}$ is composed of the following mathematical operations: at first, the one column matrices of the total acoustic pressure $\boldsymbol{P^R}$ and $\boldsymbol{P^L}$ are decomposed into the matrices of the ingoing and outgoing pressures $\boldsymbol{P^{R+}}$and $\boldsymbol{P^{R-}}$, and $\boldsymbol{P^{L+}}$and $\boldsymbol{P^{L-}}$. Similarly with the acoustic velocity, introducing the matrix of the modal admittances $\boldsymbol{Y^a}$, where $y_m = k_{z,m}/k\rho_0 c$

$$\begin{bmatrix}\boldsymbol{P^R}\\ \boldsymbol{V^R}\end{bmatrix} = \underbrace{\begin{bmatrix}\boldsymbol{T}_{11} & \boldsymbol{T}_{12}\\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22}\end{bmatrix}}_{\boldsymbol{T}} \times \begin{bmatrix}\boldsymbol{P^L}\\ \boldsymbol{V^L}\end{bmatrix} \tag{10}$$

$$(\boldsymbol{P^{R+}} + \boldsymbol{P^{R-}}) = \boldsymbol{T}_{11} \times (\boldsymbol{P^{L+}} + \boldsymbol{P^{L-}}) + \boldsymbol{T}_{12} \times \boldsymbol{Y} \times (\boldsymbol{P^{L+}} - \boldsymbol{P^{L-}}) \tag{11}$$

$$\boldsymbol{Y} \times (\boldsymbol{P^{R+}} - \boldsymbol{P^{R-}}) = \boldsymbol{T}_{21} \times (\boldsymbol{P^{L+}} + \boldsymbol{P^{L-}}) + \boldsymbol{T}_{22} \times \boldsymbol{Y} \times (\boldsymbol{P^{L+}} - \boldsymbol{P^{L-}}) \tag{12}$$

$$\boldsymbol{Y^a} = \begin{bmatrix}y_1 & & 0\\ & \ddots & \\ 0 & & y_N\end{bmatrix} \tag{13}$$

This, in turn, allows separating the expressions for the in-going and out-going waves

$$\boldsymbol{P}^{\boldsymbol{R}+} + \boldsymbol{P}^{\boldsymbol{R}-} = \underbrace{(\boldsymbol{T}_{11} + \boldsymbol{T}_{12} \times \boldsymbol{Y})}_{\boldsymbol{X}^{+}} \times \boldsymbol{P}^{\boldsymbol{L}+} + \underbrace{(\boldsymbol{T}_{11} - \boldsymbol{T}_{12} \times \boldsymbol{Y})}_{\boldsymbol{X}^{-}} \times \boldsymbol{P}^{\boldsymbol{L}-}$$
$$\boldsymbol{P}^{\boldsymbol{R}+} - \boldsymbol{P}^{\boldsymbol{R}-} = \underbrace{\boldsymbol{Y}^{-\mathbf{1}} \times (\boldsymbol{T}_{21} + \boldsymbol{T}_{22} \times \boldsymbol{Y})}_{\boldsymbol{W}^{+}} \times \boldsymbol{P}^{\boldsymbol{L}+} + \underbrace{\boldsymbol{Y}^{-\mathbf{1}} \times (\boldsymbol{T}_{21} - \boldsymbol{T}_{22} \times \boldsymbol{Y})}_{\boldsymbol{W}^{-}} \times \boldsymbol{P}^{\boldsymbol{L}-} \quad (14)$$

To simplify the notation the matrices $\boldsymbol{X}$ and $\boldsymbol{W}$ were introducited

$$\boldsymbol{P}^{\boldsymbol{R}+} + \boldsymbol{P}^{\boldsymbol{R}-} = \boldsymbol{X}^{+} \times \boldsymbol{P}^{\boldsymbol{L}+} + \boldsymbol{X}^{-} \times \boldsymbol{P}^{\boldsymbol{L}-}$$
$$\boldsymbol{P}^{\boldsymbol{R}+} - \boldsymbol{P}^{\boldsymbol{R}-} = \boldsymbol{W}^{+} \times \boldsymbol{P}^{\boldsymbol{L}+} + \boldsymbol{W}^{-} \times \boldsymbol{P}^{\boldsymbol{L}-} \quad (15)$$

what leads to the final derivation of the scattering matrix with the following sub-matrices as a result of some simple mathematical transformation

$$\boldsymbol{S}^{\mathbf{11}} = -(\boldsymbol{X}^{-} - \boldsymbol{W}^{-})^{-\mathbf{1}} \times (\boldsymbol{X}^{+} - \boldsymbol{W}^{+})$$
$$\boldsymbol{S}^{\mathbf{12}} = \mathbf{2} \cdot (\boldsymbol{X}^{-} - \boldsymbol{W}^{-})^{-\mathbf{1}}$$
$$\boldsymbol{S}^{\mathbf{21}} = \frac{1}{2}\Big((\boldsymbol{X}^{+} + \boldsymbol{W}^{+}) - (\boldsymbol{X}^{-} + \boldsymbol{W}^{-}) \times (\boldsymbol{X}^{-} - \boldsymbol{W}^{-})^{-\mathbf{1}} \times (\boldsymbol{X}^{+} - \boldsymbol{W}^{+})\Big) \quad (16)$$
$$\boldsymbol{S}^{\mathbf{22}} = (\boldsymbol{X}^{-} + \boldsymbol{W}^{-}) \times (\boldsymbol{X}^{-} - \boldsymbol{W}^{-})^{-\mathbf{1}}$$

where

$$\boldsymbol{X}^{+} = \boldsymbol{T}_{11} + \boldsymbol{T}_{12} \times \boldsymbol{Y} \qquad \boldsymbol{X}^{-} = \boldsymbol{T}_{11} - \boldsymbol{T}_{12} \times \boldsymbol{Y} \quad (17)$$

$$\boldsymbol{W}^{+} = \boldsymbol{Y}^{-\mathbf{1}} \times (\boldsymbol{T}_{21} + \boldsymbol{T}_{22} \times \boldsymbol{Y}) \qquad \boldsymbol{W}^{-} = \boldsymbol{Y}^{-\mathbf{1}} \times (\boldsymbol{T}_{21} - \boldsymbol{T}_{22} \times \boldsymbol{Y}) \quad (18)$$

Experimental eduction of the scattering matrix is especially easy applying the single mode synthetiser [13] as the necessity of calculating the invers matrix is avoided.

3. Application to the muffler description

As has been mentioned in the Introduction, for a given frequency of excitation, the number of cut-on modes depends on the radius of an element and grows with the radius. Thus, the number of propagating modes *N* can change from one element to another. This fact significantly impedes the analysis by means of the transfer matrix $\boldsymbol{T}$ because the matrix is then of a rectangular, not of a square shape and the procedure demands derivation of the reverse matrix $\boldsymbol{T}^{-\mathbf{1}}$. On the other hand, the $\boldsymbol{T}$ matrix is particularly convenient to describe the cascade system. In such a case the matrix of a whole system is a product of individual elements' matrices [1]. The scattering matrix $\boldsymbol{S}$ do not have this convenient property, but in turn it can be adopted without problems for a variable number of modes propagating in individual elements of the silencer [2, 3], as has been illustrated in Fig. 2.

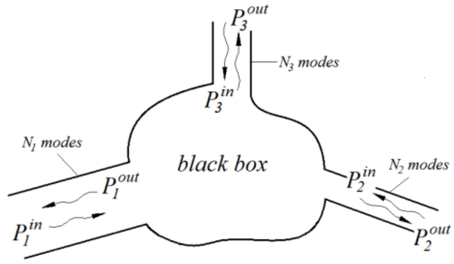

Figure 2. Scheme of a not-cascade multi-port with three connected straight ducts of different radii

The results of numerical calculations carried out according to the analytical solutions are presented below - in Fig. 3 for a muffler with one expansion chamber and in Fig. 4 – for a muffler composed of two expansion chambers. To minimise errors resulting from neglecting the effects of the near field on the junctions of the inlet/outlet pipe with the chambers, when applying the mode matching method it has been taken into account from a few up to several dozen of the excited modes, depending on the difference in radii of the joint elements. Finally, the transmission loss TL has been calculated adopting the transmission matrix (dotted and interrupted lines) and the scattering matrix method (continuous line). The numbers below the scheme of the muffler depict the relative radiuses (for example R: 1, 2, 1and R: 1, 3, 1 in Fig. 3) and length of the expansion chamber and the connecting pipe (for example L: 5, 4, 7 and L: 6, 4, 2.5 in Fig. 4).

The Helmholtz number is chosen for the inlet/outlet pipe, so up to *ka* = 1.92 in Fig. 3a and *ka* = 1.3 in Fig. 3b when the first radial Bessel mode becomes cut-on the plane wave is the only mode propagating through the muffler.

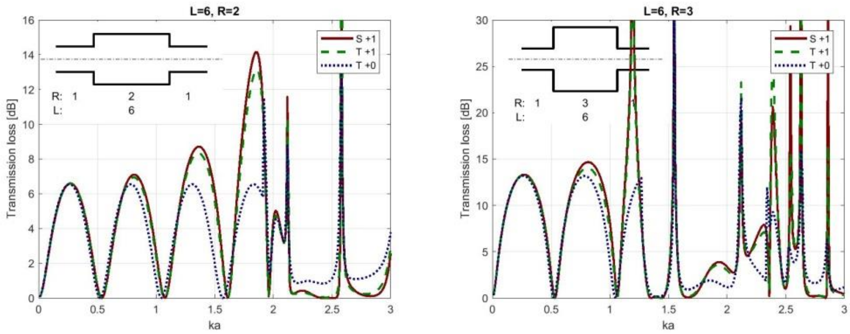

Figure 3. Transmission loss as a function of the Helmholtz number *ka* in the inlet pipe calculated by means of the transmission matrix of a size equal to the number of the cut-on modes (T+0 - dotted line) and exceeding it by one (T+1 - interrupted line) and the scattering matrix of a size exceeding by one the number of the cut-on modes (S+1 - continuous line)

However, the discrepancies between the results accounting only for the plane wave and neglecting the propagation of the attenuated first Bessel mode are visible for the Helmholtz number much smaller than these based on the cut-on frequencies. It means that having regard to the propagation of at least one cut-off mode the calculations better reflect the real muffler sound attenuation properties. According to some additional results, not presented in the paper, taking into consideration a larger number of the cut-off modes do not change the transmission loss significantly. It comes from the fact, that each of the subsequent mode is attenuated more efficiently and is extinguished practically in the vicinity of the junction.

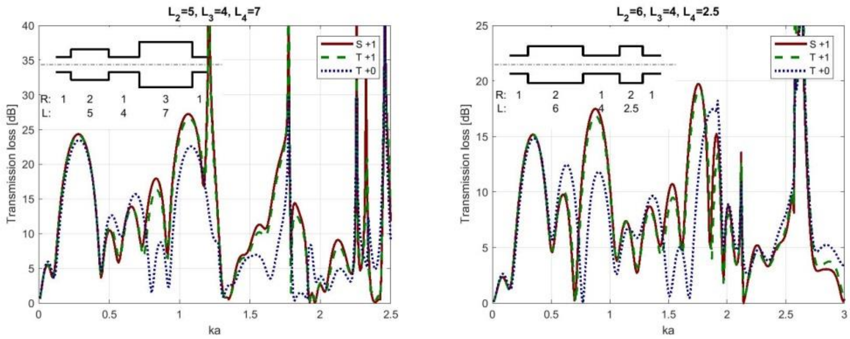

Figure 4. Transmission loss of a muffler composed of two expansion chambers as a function of the Helmholtz number *ka* in the inlet pipe calculated by means of the transmission matrix of a size equal to the number of the cut-on modes (T+0 - dotted line) and exceeding it by one (T+1 - interrupted line) and the scattering matrix of a size exceeding by one the number of the cut-on modes (S+1 - continuous line)

4. Conclusions

Research on the mufflers sound attenuation properties is still a subject of intensive scientific investigations [14] carried out by means of analytical, numerical and experimental methods. The appearance of computers with high computational power as well as the effective numerical methods has significantly intensified the numerical examination of acoustic systems with complex geometry, including silencers. However, the analytical methods are still the subject of research because of their high level of generality.

An example of a new analytical approach to the description of acoustic systems is the adaptation of the electric network method, known in acoustics as the multiport method, and within it the methods of the dispersion matrix. As shown in the article, this method is more general than the transmission matrix method commonly used in the silencer theory and can also be used for the variable number of cut-on modes in individual elements of the damper. On the other hand, the transmission matrix is particularly convenient for describing cascade systems. To sum up, the choice between the scattering and transmission matrix should depend on the property and structure of the investigated acoustic system.

References

1. M. L. Munjal, *Acoustics of ducts and mufflers*, Willey 2 ed., New York 2014.
2. A. Snakowska., K. Kolber., Ł. Gorazd, J. Jurkiewicz, Derivation of an acoustic two-port scattering matrix for a multimode wave applying the single-mode generator, *IEEE Digital Xplore Library*, (2018) 294 – 298.
3. A. Sitel, J-M Ville, F. Felix, *Multiload procedure to measure the acoustic scattering matrix of a duct discontinuity for higher order mode propagation conditions*, J. Acoust. Soc. Am., **120** (2006) 2478.
4. M. L. Munjal, *Plane wave analysis of side inlet/outlet chamber mufflers with mean flow*, Applied Acoustics, **52** (1997) 165 – 175.
5. T. Kar, M. L. Munjal. *Generalized analysis of a muffler with any number of interacting ducts*, Journal of Sound and Vibration, **285** (2005) 585 – 596.
6. C. D. Gaonkar, M. L. Munjal, *Theory of the double-tuned side-inlet side-outlet muffler*, Noise Control Engineering Journal, **66** (2018) 489 – 495.
7. T. Elnady, M. Abom, S. Allam, *Modeling perforates in mufflers using two-ports*, Journal of Vibration and Acoustics, **132** (2010).
8. M. L. Munjal, K. Narayana Rao, A. D. Sahasrabudhe, *Aeroacoustic analysis of perforated muffler components*, Journal of Sound and Vibration, **114** (1987) 173 – 188.
9. R. Glav, P. Regaudb, M. Abom, *Study of a folded resonator including the effects of higher order modes*, J. Sound. Vib., **273** (2004) 777 – 792.
10. R. Kirby, P. T. Williams, J. Hill, *A three dimensional investigation into the acoustic performance of dissipative splitter silencers*, J. Acoust. Soc. Am., **135** (2014) 2727 – 2737.
11. F. D. Denia, A. Selamet, M. J. Martínez, F. J. Fuenmayor, *Sound attenuation of a circular multi-chamber hybrid muffler*, Noise Control Engineering Journal, **56**(5) (2008) 356 – 364.
12. A. Snakowska, J. Jurkiewicz, Ł. Gorazd, *A hybrid method for determination of the acoustic impedance of an unflanged cylindrical duct for multimode wave*, J. Sound. Vib., **396** (2017) 325 – 339.
13. A. Snakowska, Ł. Gorazd, J. Jurkiewicz, K. Kolber, *Generation of a single cylindrical duct mode using a mode synthesiser*, Applied Acoustics, **114** (2016) 56 – 70.
14. M. L. Munjal, *Recent advances in muffler acoustics*, International Journal of Acoustics and Vibration, **18** (2013) 71 – 85.