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3D Dynamic Model of the Unicycle – Unicyclist System

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Abstract

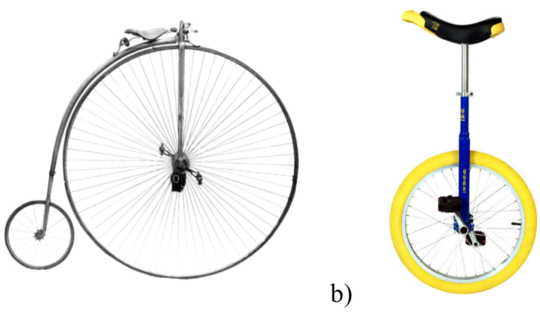
The problem of motion of a unicycle – unicyclist system in 3D is studied. The equations of motion of the system were derived using the Boltzmann-Hamel equations. Automatic generation of the Hamel coefficients eliminates all the difficulties associated with the determination of these equations. Description of the unicycle – unicyclist system dynamical model and simulation results are presented in the paper.

Keywords: unicycle, unicyclist, 3D dynamic model, Boltzmann-Hamel equations

1. Introduction

1.1. Unicycle – one wheel vehicle

Unicycle – one wheel vehicle is a specific type of single track vehicle commonly named a bicycle. It has only one road wheel. Unicycle is a "descendant" of Penny-farthing which is a bicycle with a large front driving wheel and a small rear rolling wheel. Penny-farthing and unicycle are shown in Fig. 1.



a) b)

Figure 1. a) Penny-farthing [1]; b) unicycle [2]

Unicycle was created in the late nineteenth century, after removing from penny-farthing rear wheel with frame [3]. Unicycle has considerably fewer parts than a regular bicycle. Its parts resemble counterparts for bicycles only visually.

The main feature of unicycle is fixed gear. It means that the cranks are connected rigidly to the wheels. Therefore, the rotation of the cranks directly controls the rotation of the wheel. Riding is impossible without pedalling. This gives the option to ride backwards or to stand up. Furthermore, the fixed gear imposes that the bicycle has only one ratio.

The average person standing in upright position has a centre of gravity around the belly button, as shown in Fig. 2. The unicyclist can easily maintain upright position. However, to ride comfortably, the distance between the saddle and the pedal in its lowest position must be slightly shorter than the length of the unicyclist’s leg. Thus, the centre of gravity is higher than the distance from the ground to the pedal at its lowest position. Undoubtedly, it makes riding more difficult.

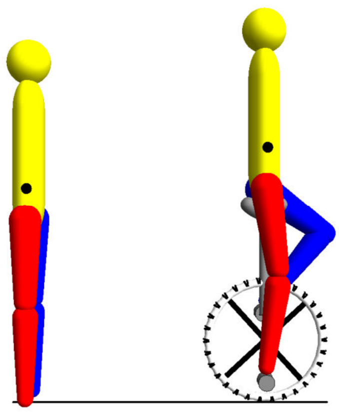


Figure 2. Average person’s centre of gravity and the unicyclist’s centre of gravity position

Riding a unicycle is more difficult than a regular bicycle, also due to the fact that there is only one point of support. For this reason, balance must be maintained in two planes simultaneously. Maintaining the balance in the plane transversal to the direction of riding (lateral) involves balancing with hips to maintain the centre of gravity above the fulcrum of the wheel. In contrast, maintaining balance in the plane parallel to the direction of travel ("forward - backward"), consists of accelerating or slowing down the drive wheel so that the centre of gravity oscillates above the fulcrum of the wheel.

1.2. Unicycle in technical aspect

In mechanical aspect unicycle with unicyclist, can be considered as a moving double inverted spherical pendulum. The first part is the unicyclist’s body, while the second, unicycle’s frame with unicyclist’s legs. If we assume that the wheel is also one of the links of the pendulum the model has the form of triple spherical pendulum. Such approach to the unicycle – unicyclist system fully corresponds to reality. It is impossible to stand in place without balancing.

During riding a unicycle the initial set value is the vertical position. Theoretically, unicyclist begins to lose balance. By measuring element, which is the membranous labyrinth feels that swings from a position of unstable equilibrium. This deflection is treated as a control error. Then, the control unit, in this case the unicyclist's brain, sends signals to appropriate parts of the body, or actuators. As the result the unicyclist balances with the whole body and returns to the upright position.

Summarizing, the unicycle with a ride-on unicyclist, in control aspect can be treated as a follow-up control system.

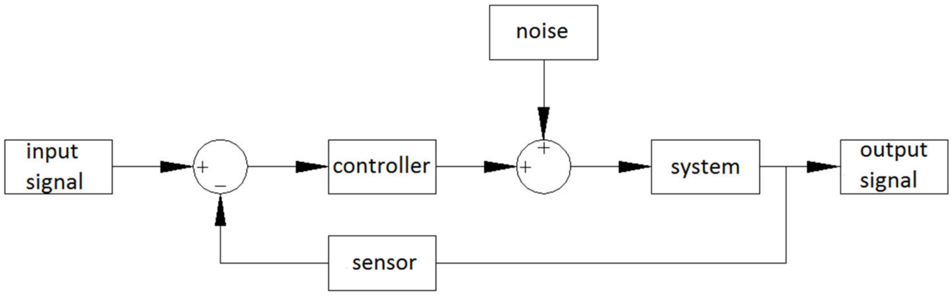


Figure 3. Follow-up control system

1.3. Boltzmann-Hamel equations

Quasi-velocities are convenient in description of motion of variable configuration systems. The introduction of quasi-velocities into the description of motion of a system is convenient when their use allows compact notation of kinetic energy and equations of motion, e.g. when investigating systems containing elements undergoing relative motion.

The Boltzmann-Hamel equations are rarely used because of complicated formulae containing Hamel coefficients (γ_{nj}^{i}) and complex relationships for the determination of these coefficients [4-8].

The classic form of the Boltzmann-Hamel equations for a system with the number of coordinates equal to k is as follows (see [4, 5])

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial w_n}\right)-\frac{\partial T^*}{\partial \pi_n}+\sum_{m=1}^{m=k}\sum_{l=1}^{l=k}\sum_{i=1}^{j=k}\sum_{j=1}^{j=k} b_{li}b_{mj}\left(\frac{\partial a_{im}}{\partial q_l}-\frac{\partial a_{il}}{\partial q_m}\right)\frac{\partial T^*}{\partial w_i}w_j=\Pi_n\,,\quad (n=1,\ldots k) \tag{1}$$

where $w_j\,(j=1,\ldots k)$ denotes quasi-velocities defined by generalized velocities $\dot{q}_n$

$$\dot{q}_n=\sum_{j=1}^{j=k} b_{nj}w_n\,,\quad (n=1,\ldots k)\,. \tag{2}$$

Introducing Hamel coefficients (γ_{nj}^{i}) defined as

$$\gamma_{nj}^{i} = \sum_{m=1}^{m=k} \sum_{l=1}^{l=k} b_{ln}\, b_{mj} \left(\frac{\partial a_{im}}{\partial q_l} - \frac{\partial a_{il}}{\partial q_m} \right), \tag{3}$$

we obtain simple form of Boltzmann-Hamel equations

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial w_n}\right) - \frac{\partial T^*}{\partial \pi_n} + \sum_{i=1}^{i=k} \sum_{j=1}^{j=k} \gamma_{nj}^{i} \frac{\partial T^*}{\partial w_i} w_j = \Pi_n, \quad (n = 1, \ldots k). \tag{4}$$

Matrix form [9] of Boltzmann-Hamel equations

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{T}^*}{\partial \mathbf{w}}\right) + \mathbf{B}^{\mathrm{T}}\left(\dot{\mathbf{A}}^{\mathrm{T}} - \mathbf{D}^{\mathrm{T}_0}\mathbf{B}\,\mathbf{w}\right)\frac{\partial T^*}{\partial \mathbf{w}} - \mathbf{B}^{\mathrm{T}} \frac{\partial \mathrm{T}^*}{\partial \mathbf{q}} = \mathbf{B}^{\mathrm{T}}\left(\mathbf{f} - \frac{\partial \mathrm{V}}{\partial \mathbf{q}}\right) \tag{5}$$

allows to automate generation of Hamel coefficients and eliminates all the difficulties associated with the determination of these quantities [9].

Analysis of dynamics of unicycle based on Boltzmann-Hamel formalism is presented in the next section.

2. Description of the analysed model

For the unicycle-unicyclist model description we use fixed inertial frame *Oxyz* (Fig. 4) as well as moving frame *Bx'y'z'* – parallel to *Oxyz*. Third frame $Bx_1y_1z_1$ is the wheel plane embedded no inertial frame.

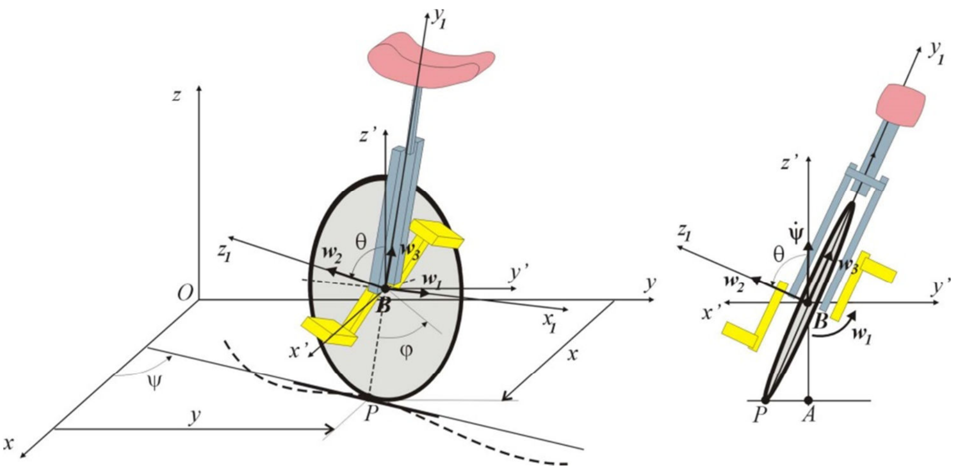


Figure 4. Model of the unicycle – without unicyclist

To consider the motion of the unicycle wheel model, we introduce the following generalized coordinates

$$\mathbf{q}_{\mathbf{w}} = [\theta, \varphi, \psi, x, y]^T, \tag{6}$$

where θ, φ, ψ are Euler angles describing wheel spatial orientation and x, y are coordinates of the contact point P. The unicycle frame orientation with respect to the wheel (frame $Bx_1 y_1 z_1$) is described by angle β_1 whereas the unicyclist upper torso orientation by angles α_2, β_2 and γ_2. The frame-unicyclist model generalized coordinates are

$$\mathbf{q}_{\mathbf{f}} = [\beta_1, \alpha_2, \beta_2, \gamma_2]^T. \tag{7}$$

Unicyclist legs orientation depends on coordinates θ, φ, ψ, β_1.

Quasi-velocities (Fig. 4) defining the wheel model velocities are assumed in the following form:

$$\begin{gathered} w_1 = \omega_1 = \dot{\theta}, \qquad w_2 = \omega_2 = \dot{\varphi} + \dot{\psi}\cos\theta, \qquad w_3 = \omega_3 = \dot{\psi}\sin\theta, \\ w_4 = -\dot{x} + a\dot{\varphi}\cos\psi, \qquad w_5 = -\dot{y} + a\dot{\varphi}\sin\psi, \end{gathered} \tag{8}$$

(a is the radius of the wheel). Equations (8) are valid under assumption that the wheel is a rigid disc making point contact with the road and rolls without longitudinal slip on a flat surface. It means that the constraint equations for the wheel are: $w_4=0$, $w_5=0$.

Matrix form of (8) defines matrices **B** and **A** used in equation (5):

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cos\theta & \cos\theta & 0 \\ 0 & 0 & \sin\theta & 0 & 0 \\ 0 & a\cos\psi & 0 & -1 & 0 \\ 0 & a\sin\psi & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \\ \dot{\psi} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{A}\,\dot{\mathbf{q}}_{\mathbf{w}}, \qquad \mathbf{B} = \mathbf{A}^{-1}. \tag{9}$$

Kinetic energy of the wheel is obtained by formula

$$T_w = \frac{1}{2} m_w a \left(w_1^2 + w_2^2\right) + \frac{1}{2}\left(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2\right). \tag{10}$$

Kinetic energy for the other parts of unicycle-unicyclist model are expressed similarly (however, these relations are more complicated).

Equations of model dynamics based on Boltzmann-Hamel equation (5) were generated automatically and solved using *Mathematica*.

3. Simulation results

Some results of numerical simulation for the unicycle-unicyclist model motion are shown in Figures 5-8.

Wheel – floor contact point trajectory and wheel centre trajectory for particular data (a = 0.3 m; m_w = 5 kg; J_1 = 0.25 kgm^2; J_2 = 0.25 kgm^2; J_3 = 0.5 kgm^2) are depicted in Fig. 5. The comparison of trajectories for different initial velocity ($\dot{\varphi}$) and roll angle (θ) are shown in Figures 6-7.

Accuracy of simulation results is shown in the Figure 8 where changes in quasi-velocities values w_4 and w_5 during simulation process are presented. Constraints equation errors have values 10^{-11} [m] (for exact solution w_4=0, w_5=0).

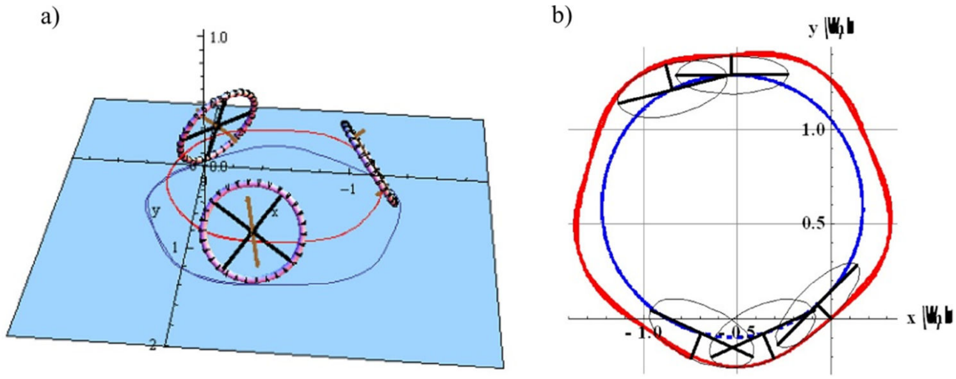


Figure 5. Unicycle wheel simulation results: a) 3D view, b) wheel – floor contact point trajectory (dashed line) and wheel centre trajectory (solid line)

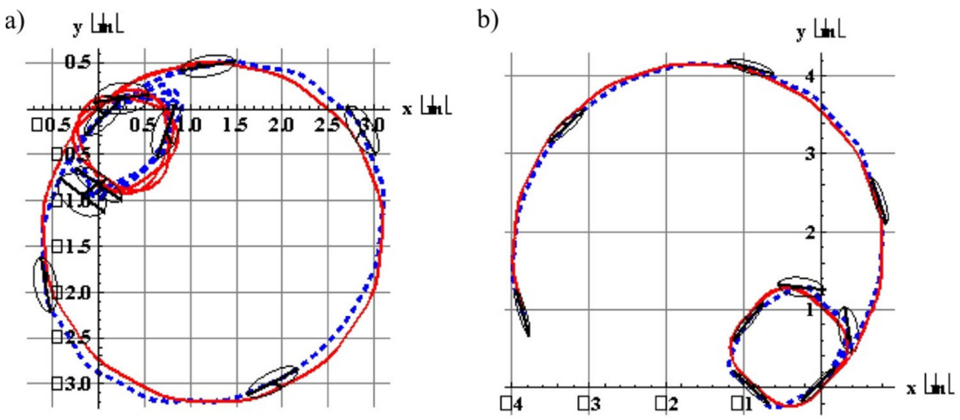


Figure 6. Comparison of contact point trajectories and wheel centre trajectory for different initial velocity ($\dot{\varphi}$) and roll angle (θ):

a) $\dot{\varphi}(0) = -3 \ rad/s$ and $\dot{\varphi}(0) = -6 \ rad/s$ (for $\theta(0) = 110^o$),

b) $\dot{\varphi}(0) = -3 \ rad/s$ and $\dot{\varphi}(0) = -5 \ rad/s$ (for $\theta(0) = 90^o$)

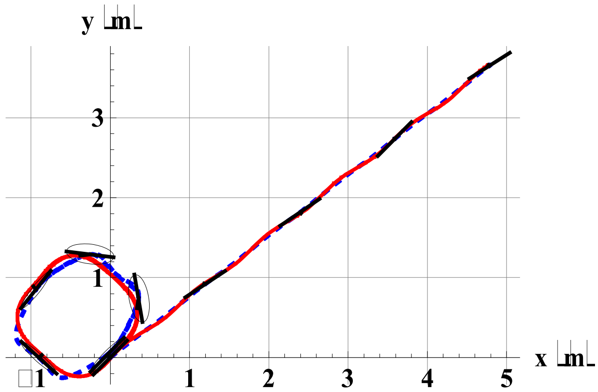


Figure 7. Comparison of contact point trajectories and wheel centre trajectory for $\dot{\varphi}(0) = -3\ rad/s$ and $\dot{\varphi}(0) = -5\ rad/s$ $(\theta(0) = 80^o)$

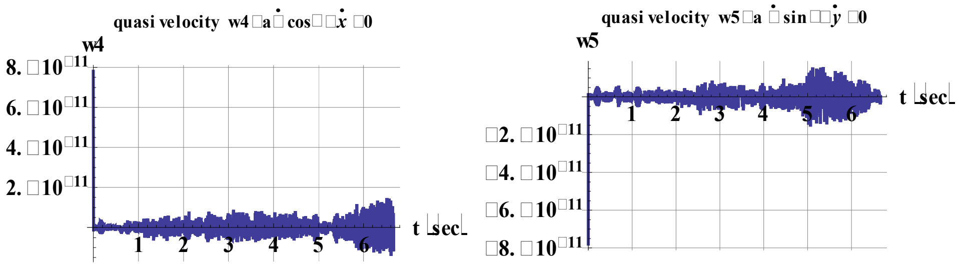


Figure 8. Constraints equations error (quasi-velocities values w_4 and w_5)

4. Conclusions

The matrix notation of Boltzmann-Hamel equations eliminates drawbacks occurring with the classical formulation of these equations. Its use allows the automation of the process of generating equations of motion. To obtain equations of motion in the form of quasi-coordinates and quasi-velocities it is sufficient to set the matrix transforming the generalized velocities into quasi-velocities, the kinetic energy and the vector of generalized forces. This procedure is general and can be used for solving many problems.

Motion analysis of a unicycle – unicyclist 3D model presented in the paper is the first step of larger work. In the future, the model will be extended and the system control method will be proposed in order to design a prototype of the unicycle, which can keep balance for unicyclist. Dynamic stability of unicycle will be also analysed.

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