An Analytical-Numerical Approach to Analysis of Large Amplitude Vibrations of Slender Periodic Beams

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Abstract

The paper is devoted to analysis of geometrically nonlinear vibrations of beams with geometric and material properties periodically varying along the axis. The 1-D Euler-Bernoulli theory of beams with von Kármán nonlinearity is applied. An analytical-numerical model based on non-asymptotic tolerance modelling approach and Galerkin method is applied. The linear natural frequencies and mode shapes are determined and the results are confirmed by comparison with a finite element model. Forced damped vibrations analysis in the large deflection range is performed to illustrate complex behaviour of the system.

Keywords: nonlinear vibrations, periodic beams, averaging, tolerance modelling

1. Introduction

Structures with physical properties arranged periodically or almost periodically in the body domain are often found in engineering and in the nature. Properly designed, they have many advantages, such as favourable mass to stiffness ratio. Furthermore, considering problems of dynamics, we can point out the frequency filtering properties of such structures, i.e. existence of frequency band gaps.

In this paper, vibrations of beams with periodically varying geometric and material properties along the longitudinal axis are considered. Equations of motion of such structures have highly oscillating, periodic, often non-continuous coefficients.

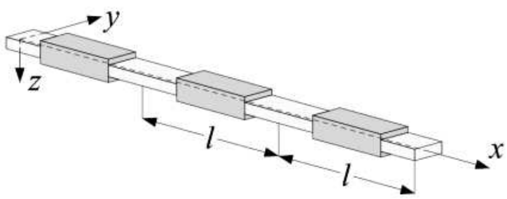

Figure 1. A fragment of a periodic beam

There are numerous special techniques in analysis of periodic media, many of them based on strict mathematical asymptotic homogenization [1]. Extensive work has been done in homogenization of periodic beams, cf. [4]. The theoretical foundation of the analytical-numerical model used here is the non-asymptotic tolerance modelling approach to analysis of microstructured periodic or almost periodic media. It is based mainly on the concepts of slowly-varying and tolerance periodic functions, and the indiscernibility relation, cf. [6]. The resulting partial differential equations with constant coefficients are then transformed into a set of ordinary differential equations using Galerkin method and then numerically integrated via the Runge-Kutta-Fehlberg method. The resulting model is an extension of a simplified one, presented in [3]. The new concept is the notion of a weakly slowly-varying function, cf. [5]. Some of the results of analysis geometrically nonlinear equilibrium problems of thin periodic plates via the tolerance modelling are confirmed in [2].

2. Equations of motion

The object under consideration is a linearly elastic, piecewise-prismatic beam. Let $Oxyz$ be an orthogonal Cartesian coordinate system, the Ox axis coincides with the axis of the beam, the cross section of the beam be symmetric with respect to the plane of the load Oxz, the load acts in the direction of the axis Oz. The beam is assumed to be made of small repetitive elements, called periodicity cells, each of which is defined as $\Delta \equiv [-l/2, l/2]$, where $l \ll L$ is the length of the cell and named the microstructure parameter.

The assumptions of the Euler-Bernoulli theory of beams with von Kármán terms serve as a basis. The effects of axial and rotational inertia are neglected, as we investigate slender elements and we are interested in analysis of transverse vibrations. Let $\partial^k = \partial^k/\partial x^k$ be the k-th derivative of a function with respect to the x coordinate. Let the transverse deflection, the longitudinal displacement, tensile and flexural stiffness, the damping coefficient, mass of the beam per unit length, transverse load and dissipative force by $w = w(x,t)$, $u_0 = u_0(x,t)$, $EA = EA(x)$, $EJ = EJ(x)$, $c = c(x)$, $\mu = \mu(x)$, $q = q(x,t)$, $p = p(x,t)$, the system of nonlinear coupled differential equations for the longitudinal displacements u_0 and the transverse deflection w can be written as:

$$\begin{aligned} &\partial^2\left(EJ\partial^2 w\right) - EA\left(\partial u_0 + \tfrac{1}{2}\left(\partial w\right)^2\right)\partial^2 w + c\dot{w} + \mu\ddot{w} = q. \\ &\partial\left[EA\left(\partial u_0 + \tfrac{1}{2}\left(\partial w\right)^2\right)\right] = 0, \end{aligned} \tag{1}$$

The coefficients EA, EJ, μ, c, and in some cases the load q, are highly oscillating, often non-continuous functions of the x coordinate.

3. Introductory concepts and basic assumptions of the tolerance modelling

To become acquainted with the basics of the method, the reader is referred to the book [6]. Here, only the fundamental concepts are presented.

Let $\Delta(x) = x + \Delta$, $\Omega_\Delta = \{x \in \Omega : \Delta(x) \subset \Omega\}$ be a cell with centre at $x \in \Omega_\Delta$. The averaging operator for an arbitrary integrable function f is defined by:

$$< f > (x) = \frac{1}{|\Delta|}\int_{\Delta(x)} f(y)dy, \quad x \in \Omega_\Delta, \quad y \in \Delta(x). \tag{2}$$

It is assumed that each of the unknown displacements w and u_0 can be decomposed into its averaged and fluctuating part, the latter of which is a finite sum of products of fluctuation shape functions (FS) and fluctuation amplitudes:

$$\begin{aligned} &w(x,t)=W(x,t)+h^A(x)V^A(x,t), \quad A=1,\ldots,N, \\ &u_0(x,t)=U(x,t)+g^K(x)T^K(x,t), \quad K=1,\ldots,M, \end{aligned} \tag{3}$$

where the functions $W(\cdot), V^A(\cdot)\in WSV_d^2(\Omega,\Delta)$, $U(\cdot), T^K \in SV_d^1(\Pi,\Delta)$ are new basic unknowns, being weakly slowly-varying or slowly-varying functions in x; the fluctuation shape functions $h^A(\cdot)\in FS_d^2(\Omega,\Delta)$, $g^K(\cdot)\in FS_d^1(\Omega,\Delta)$ are postulated *a priori* in every problem under consideration. The new basic kinematic unknowns $W(\cdot)$ and $U(\cdot)$ are called the macrodeflection and the in-plane macrodisplacements, respectively; $V^A(\cdot)$ and $T^K(\cdot)$ are additional kinematic unknowns, called the fluctuation amplitudes.

4. The averaged equations

4.1. The tolerance model

After substitution the micro-macro decomposition (3) into equations (1), the next step of modelling is averaging these equations over an arbitrary periodicity cell with weights 1, h^A and g^K. After some manipulations we arrive at the following system of equations:

$$\begin{aligned} &\langle EJ\rangle\partial^4 W+\langle EJ\partial^2 h^A\rangle\partial^2 V^A+2\langle EJ\partial h^A\rangle\partial^3 V^A+\langle EJh^A\rangle\partial^4 V^A \\ &-\langle N\rangle\partial^2 W-\langle N\partial h^A\rangle\partial V^A+\langle c\rangle\dot W+\langle\mu\rangle\ddot W+\langle ch^A\rangle\dot V^A+\langle\mu h^A\rangle\ddot V^A-\langle q\rangle=0, \end{aligned} \tag{4}$$

$$\begin{aligned} &\langle EJh^A\rangle\partial^4 W-2\langle EJ\partial h^A\rangle\partial^3 W+\langle EJ\partial^2 h^A\rangle\partial^2 W+\langle N\partial h^A\rangle\partial W+\langle\mu h^A\rangle\ddot W-\langle qh^A\rangle+ \\ &+\langle ch^A\rangle\dot W+\langle EJh^A h^B\rangle\partial^4 V^B+2\left[\langle EJh^A\partial h^B\rangle-\langle EJh^B\partial h^A\rangle\right]\partial^3 V^B+\langle ch^A h^B\rangle\dot V^B \\ &+2\left[\langle EJ\partial h^B\partial^2 h^A\rangle-\langle EJ\partial h^A\partial^2 h^B\rangle\right]\partial V^B+\langle EJ\partial^2 h^A\partial^2 h^B\rangle V^B+\langle N\partial h^A\partial h^B\rangle V^B \\ &+\left[\langle EJ\partial^2 h^B h^A\rangle+\langle EJ\partial^2 h^A h^B\rangle-4\langle EJ\partial h^A\partial h^B\rangle\right]\partial^2 V^B+\langle\mu h^A h^B\rangle\ddot V^B=0, \end{aligned} \tag{5}$$

where the averaged axial forces $\langle NF(y)\rangle$, $F(y)=\{1, \partial h^A, \partial h^A\partial h^B\}$, are independent of x:

$$\begin{aligned} \langle NF(y)\rangle = &\int_0^L V^D V^C dx\times \\ &\times\tfrac{1}{2}\left(\langle EA\partial h^C\partial h^D F(y)\rangle-\langle EA\partial g^L\rangle\langle EA\partial g^K\partial g^L\rangle^{-1}\langle EA\partial g^K\partial h^C\partial h^D F(y)\rangle\right) \\ &+\left(\langle EA\partial h^C F(y)\rangle-\langle EA\partial g^L\rangle\langle EA\partial g^K\partial g^L\rangle^{-1}\langle EA\partial g^K\partial h^C F(y)\rangle\right)\int_0^L V^C\partial W dx \\ &+\left(\langle EAF(y)\rangle-\langle EA\partial g^L\rangle\langle EA\partial g^K\partial g^L\rangle^{-1}\langle EA\partial g^K F(y)\rangle\right)\left(\Delta_0+\tfrac{1}{2}\int_0^L\partial W\partial W dx\right). \end{aligned} \tag{6}$$

Equations (4-5) with denotations (6) stand for a system of 2+N differential equations for the macrodeflection $W(\cdot)$ and for its fluctuation amplitudes $V^A(\cdot)$. As the axial inertia terms are neglected, the axial displacement $U(\cdot)$ and its fluctuation $T^K(\cdot)$ can be eliminated. The coefficients of these equations are constant, some and of them depend on the size l of the periodicity cell. Note that the elimination of axial displacement dependent terms is possible only when end displacements are restrained, but not necessarily equal to zero.

4.2. The tolerance-asymptotic model

In cases when we restrict ourselves to investigate the low frequency vibrations, we can pass with the periodicity cell length to zero, $l \to 0$. Then, some of the coefficients of equations (4)-(5) vanish. Introducing the following denotations:

$$\begin{aligned} D^{eff} &\equiv \langle EJ \rangle - \left\langle EJ\partial^2 h^A \right\rangle \left\langle EJ\partial^2 h^A \partial^2 h^B \right\rangle^{-1} \left\langle EJ\partial^2 h^B \right\rangle, \\ B^{eff} &\equiv \langle EA \rangle - \left\langle EA\partial g^K \right\rangle \left\langle EA\partial g^K \partial g^L \right\rangle^{-1} \left\langle EA\partial g^L \right\rangle, \end{aligned} \tag{7}$$

equations of the tolerance-asymptotic model take the form:

$$\begin{aligned} &D^{eff}\partial^4 W - \overline{N}\partial^2 W + C\dot{W} + M\ddot{W} - Q = 0, \\ &\overline{N} = \frac{B^{eff}}{2L}\int_0^L \partial W \partial W dx + \frac{B^{eff}}{L}\Delta_0, \quad \Delta_0 = \int_0^L \partial U dx = U(L) - U(0) \end{aligned} \tag{8}$$

The usefulness of the above formulation is restricted to analysis of long-wave modes, for which the length scale effect is not of high importance. Nevertheless, in many practically important issues such approximation is acceptable.

5. Applications

Let us investigate a piecewise-prismatic beam of length L, and periodically variable cross-section, as it is shown in Figure 2. The material of the beam is elastic and homogeneous.

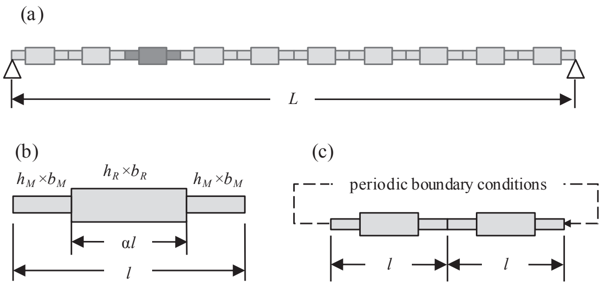

Figure 2. Scheme of the analysed beam (a), a periodicity cell (b), and periodic boundary conditions (c)

The fluctuation shape functions were obtained from a finite element analysis of a two-cell system. Each subsection of a periodicity cell was divided into two elements based on Hermite polynomials and the periodic boundary conditions were assumed, as indicated in Figure 2(c). The obtained mode shapes can be divided into two groups of even (*ESF*) and odd (*OSF*) shape functions, cf. Figure 3.

The solutions to the tolerance model and the load were assumed as finite sums:

$$\begin{Bmatrix} W(x,t) \\ Q(x,t) \end{Bmatrix} = \sum_{m=1}^{M_w} \begin{Bmatrix} W_m(t) \\ Q_m(t) \end{Bmatrix} X_m(x), \quad \begin{Bmatrix} V^A(x,t) \\ Q^A(x,t) \end{Bmatrix} = \sum_{n=1}^{M_V^A} \begin{Bmatrix} V_n^A(t) \\ Q_n^A(t) \end{Bmatrix} Y_n^A(x), \quad A = 1, \ldots, N, \tag{9}$$

where the functions X_m and Y^A_n were chosen to satisfy the boundary conditions of a simply supported beam:

$$X_m(x) = \sin(m\pi x/L), \quad Y_n^A(x) = \begin{cases} \sin(n\pi x/L) & \text{for } A \in ESF, \\ \cos[(n-1)\pi x/L] & \text{for } A \in OSF. \end{cases} \tag{10}$$

That leads to the following system ordinary differential equations of second order:

$$[\mathbf{K}_0 + \mathbf{K}_{NL}(\mathbf{y})]\mathbf{y} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{M}\ddot{\mathbf{y}} = \mathbf{f}. \tag{11}$$

After dropping the nonlinear, damping and forcing terms the linear natural frequencies and mode shapes are determined from analysis of the generalized eigenvalue problem. The results of comparison with a full finite element model of a beam are presented in Section 5.1.

Then, the nonlinear model based on the asymptotic approximations (8) is applied in analysis of damped forced vibrations. It is justified only when the forcing frequency is of the order of the few lowest natural frequencies of the beam. The analysed equations and used denotations are given by formulas (12) and (13), respectively.

$$\begin{aligned} &\omega_m{}^2 w_m(t) + \ddot{w}_m(t) + 2\beta \dot{w}_m(t) + \\ &\qquad + \gamma_m \sum_n n^2 w_n(t)^2 w_m(t) + P_m w_m(t) - p_m f_0 \cos\Omega t = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &\omega_m = \left(\frac{m\pi}{L}\right)^2 \sqrt{\frac{D^{eff}}{M}}, \quad \gamma_m = \frac{m^2}{4}\left(\frac{\pi}{L}\right)^4 \frac{B^{eff}}{M}, \\ &P_m = \left(\frac{m\pi}{L}\right)^2 \frac{B^{eff}}{M}\frac{\Delta_0}{L}, \quad \beta = \frac{C}{2M}, \quad p_m = \frac{q_m}{M}. \end{aligned} \tag{13}$$

The results of analysis are briefly described in Section 5.2.

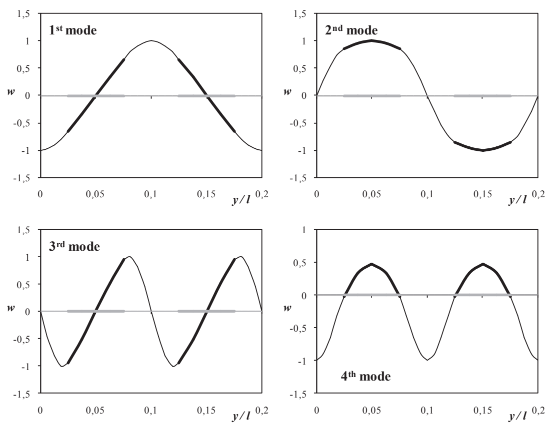

Figure 3. The first four modes of a two-cell system used as fluctuation shape functions

5.1. Natural linear frequencies and mode shapes

The object of this section is to perform a limited confidence check of the model accuracy. The analysed beam (cf. Figure 2) has length $L = 1.0$ m, the elastic modulus is $E = 205$ GPa, the mass density $\rho = 7850$ kg/m3. The cross section is rectangular: $b_M = b_R = 10$ mm, $h_M = 5$ mm, $h_R = 10$ mm, other geometric parameters of the cell are $l = 1/10$ m, $\alpha = 1/2$.

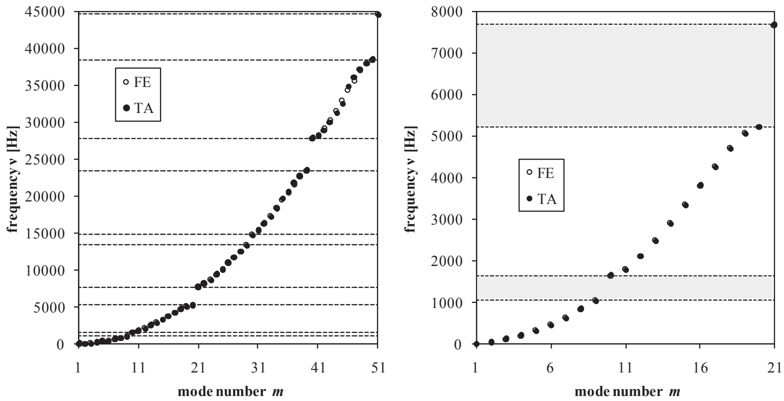

Figure 4. Comparison of first 51 (left) and first 21 (right) natural frequencies obtained from tolerance (closed circles) and finite element model (open circles)

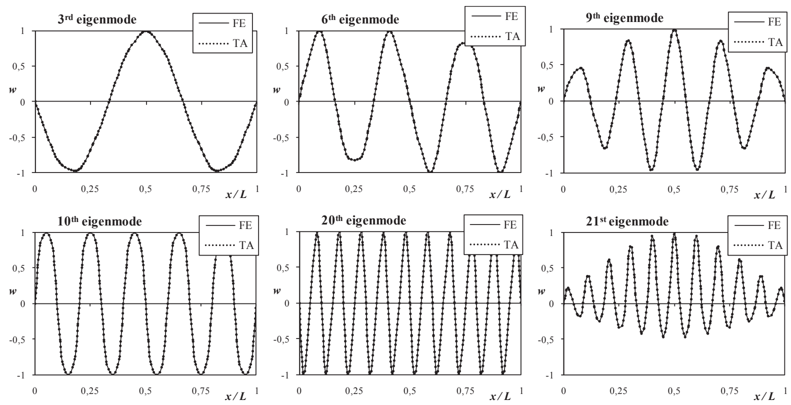

Figure 5. Comparison of chosen natural modes of considered beam obtained from tolerance (TA - dotted lines) and finite element (FE - solid lines) model.

The first four of 23 modes of a two-cell assemblage used as fluctuation shape functions are shown in Figure 3. For comparison, a finite element model of the full beam has been formulated. The natural frequencies and mode shapes were determined from the equation det($\mathbf{K}_0$ - $\omega^2\mathbf{M}$)=0, cf. (11). Figure 4 presents the comparison between tolerance modelling (TA) and finite element (FE) results for first 51 frequencies and its close-up in the range of first 21 frequencies, where the lower band-gaps are more visible. The 3rd, 6th, 9th, 10th, 20th and 21st natural modes obtained from both models are depicted in Figure 5. The results are in good agreement. It has to be mentioned that all the upper and lower boundaries of band gaps correspond to the first (n=1) modes of fluctuation amplitudes, cf. relationships (9) and (10). The proposed model gives satisfactory results not only in the low frequency range.

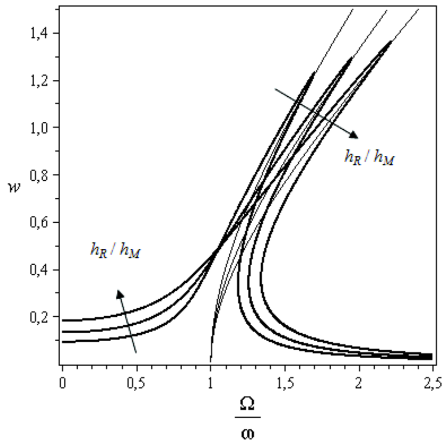

Figure 6. Backbone and amplitude-frequency curves

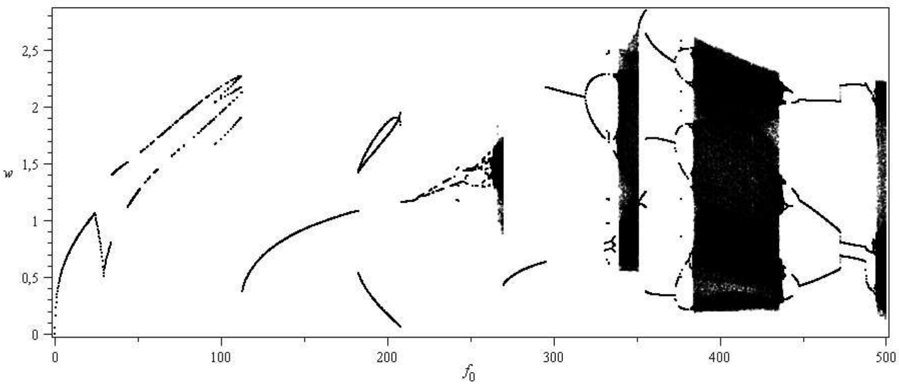

Figure 7. Bifurcation diagram of central deflection w versus forcing amplitude f_0

5.2. Nonlinear vibrations analysis

Let us consider a problem of forced damped vibrations of a beam introduced in the beginning of this section, governed by the equations (12). The material and geometric parameters remain the same, although three cases were considered here: a) $\alpha = 4/5$, $h_R / h_M = 13/8$; b) $\alpha = 1/2$, $h_R / h_M = 2$; c) $\alpha = 1/5$, $h_R / h_M = 3$. That is, the total mass of the beam is kept constant, but the effective bending and axial stiffness is: $D^{eff} = \{55.259;\ 37.963;\ 26.538\}$ Nm^2, and $B^{eff} = \{1.481;\ 1.367;\ 1.196\}\times 10^7$ N, and the first natural linear frequencies are $\omega_1 = \{95.617;\ 79.253;\ 66.263\}$ rad/s. The coefficient of the external damping was assumed to be $c = 2.5$ Ns/m.

First, the one-term approximation of to the equations (12) has been used to determine the backbone curves and amplitude-frequency response curves shown in Figure 6. Light forcing amplitude ($f_0 = 4.25$) and forcing frequency near the fundamental frequency was assumed. Next, five-term approximation to these equations has been applied in analysis of long-term forced vibrations for case (b). The forcing frequency is equal to the first natural frequency of the beam. The bifurcation diagram with forcing amplitude f_0 as a parameter is displayed in Figure 7. Complicated behaviour of the system is exposed, including periodic oscillations, symmetry breaking and saddle-node bifurcations, as well as period-doubling routes to chaos. More detailed analysis of the results will be presented and discussed in forthcoming papers.

6. Conclusions

It can be concluded that the presented model properly describes the crucial dynamic characteristics of beams with periodic structure and it can be used as a reliable tool in parametric analysis of vibration problems. The advantage of proposed approach is that it allows for the construction of models of low degree of freedom number.

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