Vibrations of Non-Periodic Thermoelastic Laminates

Jarosław JĘDRYSIAK
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, jarek@p.lodz.pl

Ewelina PAZERA
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, ewelina.pazera@p.lodz.pl

Abstract

Vibrations of non-periodic thermoelastic laminates, which can be treated as made of functionally graded material with macroscopic properties changing continuously along direction, x_1, perpendicular to the laminas on the macrolevel are considered. Three models of these laminates are presented: the tolerance, the asymptotic-tolerance and the asymptotic. Governing equations of two first of them involve terms dependent of the microstructure size. Hence, these models (the tolerance, the asymptotic-tolerance) describe the effect of the microstructure. Averaged governing equations of these laminates can be obtained using the tolerance modelling technique, cf. Jędrysiak [1]. Because the model equations have functional, but slowly-varying coefficients calculations for examples can be made numerically or by using approximated methods.

Keywords: nonperiodic laminates, thermoelasticity, vibrations, microstructure, tolerance modelling

1. Introduction

The objects under consideration are non-periodic laminates, made of two components, which are non-periodically distributed along a direction normal to laminas. Cells of them are composed of two sublaminas of different materials. Macroscopic properties of these laminates are assumed to be continuously varied along this direction, cf. Figure 1. A microstructure can be realised as uniform, l=const, or non-uniform, l=$l(x)$, distribution of laminas (Figures 1b, 1c), cf. Jędrysiak [2]. Hence, these laminates can be called *transversally* or *functionally graded laminates*, cf. Jędrysiak and Radzikowska [3].

Although a microstructure of these laminates is not periodic, thermomechanical problems of them can be investigated using micromechanical models proposed for composites with idealised geometries, e.g. periodic. Hence, the behaviour of these media can be analysed by certain modified methods, which are also applied to macroscopically homogeneous composites. Some of these methods are explained by Suresh and Mortensen [4] or Reiter et al. [5]. Between them techniques based on the asymptotic homogenization, [6], or on concepts of microlocal parameters, [7], can be mentioned. Various alternative approaches are proposed to describe the behaviour of functionally graded materials, such as the higher-order theory shown by Aboudi et al [8]. Unfortunately, governing equations of most of these approaches neglect *the effect of the microstructure size* on the overall behaviour of these laminates.

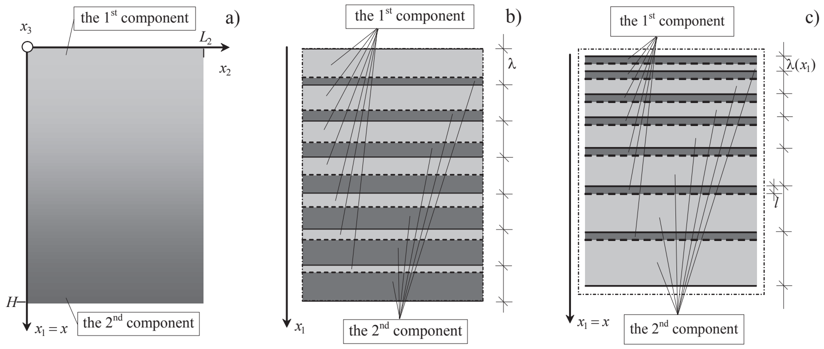

Figure 1. A part of the laminate: a) the macro-level, b) the micro-scale with uniform distribution of laminas, c) the micro-scale with non-uniform distribution of laminas; [2]

Here, in order to describe this effect the tolerance modelling is applied, cf. the books by Woźniak and Wierzbicki [9], edited by Woźniak et al. [10, 11] and by Jędrysiak [1].

This method was proposed and used to investigate different thermomechanical problems of periodic media, e.g. for thermoelastic processes by Ignaczak [12] or Baczyński [13]. Examples of analysis various periodic structures can be found in [10]. Moreover, the tolerance modelling is successfully used to investigate thermomechanical problems of functionally graded media with a microstructure in a series of papers, e.g. for vibrations of thin microstructured plates by Jędrysiak [1]; for heat conduction problems by Ostrowski and Michalak [14], Jędrysiak and Radzikowska [3], Jędrysiak [2]; for thermoelasticity problems by Jędrysiak [1, 15], Pazera and Jędrysiak [16]. All these problems are described for FG-type structures by differential equations with highly oscillating, tolerance-periodic, non-continuous, functional coefficients. The tolerance modelling leads from these equations to the system of differential equations with slowly-varying coefficients. Some applications of this approach for transversally graded structures are also shown in books by Jędrysiak [1], Michalak [17].

The main aim is to present and apply the governing equations of the tolerance model, the asymptotic-tolerance model and the asymptotic model to the problem of vibrations of a functionally graded laminated layer. The equations of two the first aforementioned models (the tolerance and the asymptotic-tolerance) involve terms, which describe the effect of the microstructure size on the overall behaviour of these laminates.

2. Modelling foundations

Denote by $Ox_1x_2x_3$ the orthogonal Cartesian coordinate system and by t the time coordinate. Let: $\mathbf{x}\equiv(x_2,x_3)$, $x\equiv x_1$. The region of the undeformed laminate is described by $\Omega\equiv(-L/2,L/2)\times(-L_2/2,L_2/2)\times(-L_3/2,L_3/2)$, with the lengths L, L_2, L_3 along the x, x_2-, x_3-axis, repectively. The "basic cell" $\Delta\equiv[-l/2,l/2]$ is defined in the interval $\Lambda=(-L/2,L/2)$ along the x-axis, with l as the length of cell Δ, called *the microstructure parameter*. Parameter l is assumed to satisfy the condition $l<<L$.

Denote by c_{ijkl}, ρ, b_{ij}, k_{ij}, c elasticity modulus, a mass density, thermoelasticity modulus, heat conduction coefficients, a specific heat, respectively, which can be assumed to be highly-oscillating, non-continuous functional coefficients of x. Introduce displacements u_i (i,j,k,l=1,2,3) and temperature θ.

Thermoelasticity problems of composites can be describe by the following equations:

$$\begin{aligned} &\partial_j(c_{ijkl}\partial_l u_k) - \rho\ddot{u}_i = \partial b_{ij}\theta + b_{ij}\partial_j\theta, \\ &\partial_j(k_{ij}\partial_i\theta) = c\dot{\theta} + T_0 b_{ij}\partial_j\dot{u}_i, \end{aligned} \tag{1}$$

which have highly-oscillating, tolerance-periodic, non-continuous coefficients being functions in x.

3. Modelling concepts

Some basic concepts, defined in books [1, 10-11], are applied in the tolerance modelling.

Denote $\Delta(x)\equiv x+\Delta$, $\Lambda_\Delta=\{x\in\Lambda: \Delta(x)\subset\Lambda\}$, as a cell at $x\in\Lambda_\Delta$. The first concept is *the averaging operator* for an arbitrary integrable function f, defined by

$$< f > (x, x_2) = \tfrac{1}{l}\int_{\Delta(x)} f(y, x_2)dy, \quad x\in\Lambda_\Delta. \tag{2}$$

Averaged value of function f being tolerance-periodic in x, calculated by (2) is a slowly-varying function in x.

Following [1, 10, 11] more introductory concepts are introduced and applied: tolerance-periodic functions $TP_\delta^1(\Lambda,\Delta)$, slowly-varying functions $SV_\delta^1(\Lambda,\Delta)$, highly oscillating functions $HO_\delta^1(\Lambda,\Delta)$, with δ as a tolerance parameter. *The fluctuation shape function* $g(\cdot)\in FS_\delta^1(\Lambda,\Delta)$, is a very important concept, which is a continuous highly oscillating function, dependent on l; has a piecewise continuous and bounded gradient $\partial^1 g$; satisfies conditions: $g\in O(l)$, $\partial^1 g\in O(l^0)$; $<\mu g>(x)\approx 0$ for $x\in\Lambda_\Delta$, $\mu>0$, $\mu\in TP_\delta^1(\Lambda,\Delta)$.

4. The outline of the modelling procedures

The various modelling procedures based on the concepts of the tolerance modelling are shown in the books [1, 11]. Here, the outline of them is presented.

- *The outline of the tolerance modelling procedure*

Two fundamental assumptions are formulated in the tolerance modelling procedure. The first assumption of them is *the micro-macro decomposition*, where the displacements and the temperature are decomposed as:

$$u_i(x,\mathbf{x},t) = w_i(x,\mathbf{x},t) + h(x)v_i(x,\mathbf{x},t), \qquad \theta(x,\mathbf{x},t) = \vartheta(x,\mathbf{x},t) + g(x)\psi(x,\mathbf{x},t), \tag{3}$$

with new basic unknowns: *the macrodisplacements* w_i, *the macrotemperature* ϑ, and *the fluctuation amplitudes of displacements* v_i, and *temperature* ψ, which all of them are slowly-varying functions in x; $h(x)$, $g(x)$ are the known fluctuation shape functions, assumed here as saw-like functions.

The tolerance averaging approximation is the second assumption, in which it is assumed that terms of an order of $O(\delta)$ are negligibly small, cf. [1, 10, 11], e.g. in: $< f\partial_1(gF) > (x) = < f\partial_1 g > (x)F(x) + O(\delta)$, $< fF > (x) = < f > (x)F(x) + O(\delta)$, for $f\in TP_\delta^1(\Lambda,\Delta)$, $g\in FS_\delta^1(\Lambda,\Delta)$, $F\in SV_\delta^1(\Lambda,\Delta)$.

Substituting micro-macro decompositions (3) to governing equations (1), by doing averaging (2), after some manipulations the governing equations of the averaged models can be derived.

- *The outline of the asymptotic-tolerance modelling procedure*

This modelling procedure, cf. [1, 11], can be divided into two steps. The first step is to apply the asymptotic modelling approach to obtain the asymptotic model solutions in the form:

$$u_{0i}(x,\mathbf{x},t)=w_i(x,\mathbf{x},t)+h(x)v_i(x,\mathbf{x},t), \qquad \theta_0(x,\mathbf{x},t)=\vartheta(x,\mathbf{x},t)+g(x)\psi(x,\mathbf{x},t). \tag{4}$$

It is derived the system of differential equations only for the macrodisplacements and the macrotemperature. In the second step there are introduced the additional micro-macro decompositions to these equations,:

$$u_i(x,\mathbf{x},t)=u_{0i}(x,\mathbf{x},t)+f(x)r_i(x,\mathbf{x},t), \qquad \theta(x,\mathbf{x},t)=\theta_0(x,\mathbf{x},t)+d(x)\chi(x,\mathbf{x},t), \tag{5}$$

with functions: w_i, v_i, ϑ, ψ (known from the asymptotic model solution); new unknown slowly-varying functions: r_i, χ; fluctuation shape functions f, d similar to h, g.

Using these modelling procedures, shown explicitly in [1, 11], the equations of the tolerance model, the asymptotic model and the asymptotic-tolerance model for functionally graded laminates can be derived. These model equations are written in the next section.

5. Model governing equations

Hence, the tolerance modelling procedure, cf. [1, 11, 15], leads to the system of governing equations in the following form:

$$\begin{gathered}
\partial_j(<c_{ijkl}>\partial_l w_k+<c_{ijk1}\partial h>v_k)-<\rho>\ddot{w}_i=\partial<b_{i1}>\vartheta+<b_{ij}>\partial_j\vartheta,\\
-\underline{c_{i\alpha k\beta}hh>\partial_\alpha\partial_\beta v_k}+<c_{i1k1}\partial h\partial h>v_k+<c_{i1kl}\partial h>\partial_l w_k+\underline{<\rho hh>\ddot{v}_i}=\\
=-<b_{i1}\partial g>\vartheta+\underline{<b_{i\beta}gh>\partial_\beta\psi},\\
\partial_j(<k_{ij}>\partial_i\vartheta+<k_{1j}\partial g>\psi)=<c>\dot{\vartheta}+<T_0 b_{ij}>\partial_j\dot{w}_i+<T_0 b_{i1}\partial h>\dot{v}_i,\\
\underline{<k_{\alpha\beta}gg>\partial_\alpha\partial_\beta\psi}-<k_{i1}\partial g>\partial_i\vartheta-<k_{11}\partial g\partial g>\psi=\underline{<cgg>\dot{\psi}}+\underline{<T_0 b_{i\beta}hg>\partial_\beta\dot{v}_i},
\end{gathered} \tag{6}$$

with all coefficients being slowly-varying functions in x. These equations together with micro-macro decompositions (3) determine *the tolerance model of thermomechanics of functionally graded laminates*. The underlined terms depend on the microstructure parameter l. Hence, equations (6) describe the effect of the microstructure size of these laminates. The basic unknowns w_i, v_i, ϑ, ψ, i=1,2,3, are slowly-varying functions in x. It can be observed that boundary conditions have to be formulated for *the macrodisplacements* w_i and *the macrotemperature* ϑ on all edges, and for *the fluctuation amplitudes* v_i, ψ only for edges normal to the x_2- and the x_3-axis.

Using the asymptotic-tolerance modelling procedure, cf. [1, 11], governing equations take the form:

$$\begin{gathered}
\partial_j(<c_{ijkl}>\partial_l w_k+<c_{ijk1}\partial h>v_k)-<\rho>\ddot{w}_i=\partial<b_{i1}>\vartheta+<b_{ij}>\partial_j\vartheta,\\
\underline{<c_{i\alpha k\beta}ff>\partial_\alpha\partial_\beta r_k}-\underline{<\rho ff>\ddot{r}_i}-<c_{i1k1}\partial f\partial f>r_k=<c_{i1k1}\partial h\partial f>v_k+<c_{i1kl}\partial f>\partial_l w_k+<b_{i1}\partial f>\vartheta,\\
\partial_j(<k_{ij}>\partial_i\vartheta+<k_{1j}\partial g>\psi)=<c>\dot{\vartheta}+<T_0 b_{ij}>\partial_j\dot{w}_i+<T_0 b_{i1}\partial h>\dot{v}_i,\\
\underline{<k_{\alpha\beta}dd>\partial_\alpha\partial_\beta\chi}-<k_{11}\partial d\partial d>\chi=<k_{i1}\partial d>\partial_i\vartheta+<k_{11}\partial g\partial d>\psi+\underline{<cdd>\dot{\chi}}+\underline{<T_0 b_{i\beta}fd>\partial_\beta\dot{r}_i},\\
<c_{i1k1}\partial h\partial h>v_k=-<c_{i1kl}\partial h>\partial_l w_k-<b_{i1}\partial g>\vartheta, \qquad <k_{11}\partial g\partial g>\psi=-<k_{i1}\partial g>\partial_i\vartheta.
\end{gathered} \tag{7}$$

These equations have smooth, slowly-varying coefficients in the contrast to equations (1). Equations (7) together with micro-macro decompositions (4)-(5) stand *the asymptotic-tolerance model of thermomechanics of functionally graded laminates*. These equations take into account the effect of the microstructure size of these laminates, since the underlined terms depend on the microstructure parameter l. The basic unknowns w_i, r_i, ϑ, χ, i=1,2,3, are slowly-varying functions in x. It can be observed that boundary conditions have to be formulated for *the macrodisplacements* w_i and *the macrotemperature* ϑ on all edges, and for *the fluctuation amplitudes* v_i, ψ only for edges normal to the x_2- and the x_3-axis.

Using the asymptotic modelling procedure, cf. [1, 11], the following governing equations can be derived:

$$\begin{aligned}
&\partial_j(<c_{ijkl}>\partial_l w_k+<c_{ijk1}\partial h>v_k)-<\rho>\ddot{w}_i=\partial<b_{i1}>\vartheta+<b_{ij}>\partial_j\vartheta,\\
&\partial_j(<k_{ij}>\partial_i\vartheta+<k_{1j}\partial g>\psi)=< c >\dot{\vartheta}+<T_0 b_{ij}>\partial_j\dot{w}_i+<T_0 b_{i1}\partial h>\dot{v}_i,\\
&<c_{i1k1}\partial h\partial h>v_k=-<c_{i1kl}\partial h>\partial_l w_k-<b_{i1}\partial g>\vartheta,\\
&<k_{11}\partial g\partial g>\psi=-<k_{i1}\partial g>\partial_i\vartheta.
\end{aligned} \tag{8}$$

The above equations have smooth, slowly-varying coefficients and together with micro-macro decompositions (4) determine *the asymptotic model of thermomechanics of functionally graded laminates*. These equations neglect the effect of the microstructure size of these laminates. It can be observed that boundary conditions have to be formulated for *the macrodisplacements* w_i and *the macrotemperature* ϑ on all edges. The asymptotic model equations describe thermoelasticity of these laminates on the macrolevel only.

6. Remarks

In this note three systems of averaged governing equations of functionally graded laminates are shown. These equations are derived using different modelling procedures – the tolerance modelling, the asymptotic modelling and a combination of them – the asymptotic-tolerance modelling. These procedures lead from the governing equations of thermoelasticity in laminates, with coefficients being non-continuous, tolerance-periodic functions in x to the systems of differential equations having slowly-varying coefficients of x for each model.

Two of presented models – *the tolerance* and *the asymptotic-tolerance*, make it possible to analyse the effect of the microstructure size in thermoelasticity problems of these laminates. Both of these models can describe not only macrovibrations, but also microvibrations, related to the microstructure of the functionally graded laminates.

However, *the asymptotic model*, since its model equations neglect the above effect, describes only macrovibrations of these composites.

Because the equations of all models have still functional coefficients, but slowly-varying, solutions of them can be found analytical only for special cases of distribution of properties of laminates, or using approximate methods. It will be shown in forthcoming papers.

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