Higher Order Vibrations of Thin Periodic Plate Bands with Various Boundary Conditions

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Abstract

In this contribution there are considered thin periodic plates. The tolerance averaging method, cf. [12, 13, 4], is applied to model problems of vibrations of these plates. Hence, the effect of the microstructure size is taken into account in model equations of the tolerance model. Calculations are made for periodic plate bands using this model and the asymptotic model for various boundary conditions.

Keywords: periodic plates, effect of microstructure size, higher order vibrations, tolerance modelling

1. Introduction

Thin periodic plate bands are main objects under consideration. These plate bands have a periodic microstructure along their spans on the microlevel, cf. Figure 1.

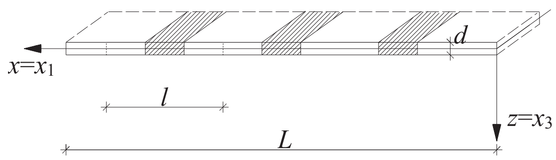

Figure 1. Fragment of a thin periodic plate band

Plate bands of this kind are consisted of many repeated small elements. Every element can be treated as a thin plate band with span l along the x_1-axis. This span describes the size of the microstructure and is called *the microstructure parameter l*. It is necessary to distinguish that in various problems of such plate bands *the effect of the microstructure size* cannot be neglected. These plates are modelled using different averaging approaches, e.g. based on the asymptotic homogenization, cf. [7]. However, most averaged equations of these plates neglect the effect of the microstructure size.

In order to take into account this effect *the tolerance averaging technique*, cf. [12] and [13], can be applied. Different applications of this method to analyse various periodic structures are shown in a series of papers, e.g. [1-3], [8-11]. This approach is also successfully adopted to functionally graded structures, e.g. [4-6].

The main aim of this note is to present governing equations of the *tolerance model* and the *asymptotic model* of thin periodic plates. Equations of these models can be derived using the tolerance modelling procedure and the asymptotic modelling procedure, respectively. In an example there are analysed lower and higher free vibration frequencies of periodic plate bands with various boundary conditions.

2. Modelling foundations

Set $\mathbf{x}\equiv(x_1,x_2)$, $x\equiv x_1$, $z\equiv x_3$. Let us consider a periodic plate band with span L along the x-axis. Hence, all properties of the plate can be periodic functions of x, but are independent of x_2. Denote a plate deflection by $w(x,t)$, loads normal by p and a derivative with respect to x by $\partial(\cdot)$. The region $\Omega\equiv\{(x,z):-d(x)/2\leq z\leq d(x)/2, x\in\Lambda\}$ denotes the undeformed plate band, with an interval $\Lambda=[0,L]$ and the plate thickness $d(\cdot)$. The *periodicity cell* on Λ is denoted by $\Delta\equiv[-l/2,l/2]\times\{0\}$.

Properties of the plate band are determined by periodic functions of x: a mass density per unit area μ, a rotational inertia ϑ and bending stiffnesses $b_{\alpha\beta\gamma\delta}$ in the form:

$$\mu(x)\equiv\int_{-d/2}^{d/2}\rho(x,z)dz,\qquad \vartheta(x)\equiv\int_{-d/2}^{d/2}\rho(x,z)z^2dz,\qquad b_{\alpha\beta\gamma\delta}(x)\equiv\int_{-d/2}^{d/2}c_{\alpha\beta\gamma\delta}(x,z)z^2dz. \tag{1}$$

Denoting $b\equiv b_{1111}$ and using the Kirchhoff-type plates theory assumptions the known four order differential equation for deflection $w(x,t)$ of periodic plate band can be derived:

$$\partial\partial(b\partial\partial w)+\mu\ddot{w}-\partial(\vartheta\partial\ddot{w})=p, \tag{2}$$

with highly oscillating, periodic, non-continuous coefficients being functions of x.

3. The outline of the tolerance modelling

Averaged equations thin periodic plates can be obtained using the tolerance modelling procedure (or the asymptotic procedure), with the basic concepts, defined in books, cf. [12, 13, 4].

Let $\Delta(x)\equiv x+\Delta$, $\Lambda_\Delta=\{x\in\Lambda\colon \Delta(x)\subset\Lambda\}$, be a cell at $x\in\Lambda_\Delta$. The averaging operator for an arbitrary integrable function f is defined by

$$<f>(x)=\frac{1}{l}\int_{\Delta(x)}f(y)dy,\quad x\in\Lambda_\Delta. \tag{3}$$

If a function f is periodic in x, then averaged value by (3) is constant.

Following the above books there can be introduced a set of tolerance-periodic functions $TP_\delta^\alpha(\Lambda,\Delta)$, a set of slowly-varying functions $SV_\delta^\alpha(\Lambda,\Delta)$, a set of highly oscillating functions $HO_\delta^\alpha(\Lambda,\Delta)$, ($\alpha\geq0$, δ is a tolerance parameter). Denote by $h(\cdot)$ a continuous highly oscillating function, $h\in FS_\delta^2(\Lambda,\Delta)$. Function $h(\cdot)$ is called *the fluctuation shape function* of the 2-nd kind, if it depends on l as a parameter and satisfies conditions: $\partial^k h\in O(l^{\alpha-k})$ for $k=0,1,\ldots,\alpha$, $\partial^k h\equiv h$, and $<\mu h>(x)\approx0$ for every $x\in\Lambda_\Delta$, $\mu>0$, $\mu\in TP_\delta^1(\Lambda,\Delta)$.

Using the above concepts, two fundamental assumptions of the tolerance modelling can be formulated, cf. Woźniak et al. [12, 13] and for thin periodic plates [3].

The first assumption is *the micro-macro decomposition*, in which it is assumed that the plate deflection can be decomposed as:

$$w(x,z,t)=W(x,t)+h^A(x)V^A(x,t),\qquad A=1,\ldots,N. \tag{4}$$

Functions $W(\cdot,t), V^A(\cdot,t) \in SV_\delta^2(\Lambda,\Delta)$ are basic kinematic unknowns, called *the macrodeflection* and *the fluctuation amplitudes*, respectively; $h^A(\cdot)$ are the known *fluctuation shape functions*, which can be assumed as trigonometric functions.

The second assumption is *the tolerance averaging approximation*, i.e. terms of an order of $O(\delta)$ can be treated as negligibly small, cf. [12, 13, 3], e.g. for $f \in TP_\delta^2(\Lambda,\Delta)$, $h \in FS_\delta^2(\Lambda,\Delta)$, $F \in SV_\delta^2(\Lambda,\Delta)$, in: $<f>(x) = <\bar{f}>(x) + O(\delta)$, $<fF>(x) = <f>(x)F(x) + O(\delta)$, $<f\partial(hF)>(x) = <f\partial h>(x)F(x) + O(\delta)$.

The tolerance modelling procedure can be found in the books [12, 13, 4]. Here, it is shown only an outline of this method.

In the tolerance modelling two basic steps can be introduced. In the first step micro-macro decomposition (4) is applied. In the second step averaging operator (3) is used to the resulting formula. Hence, the tolerance averaged lagrangean $<\Lambda_h>$ is obtained:

$$
\begin{aligned}
<\Lambda_h> = -\tfrac{1}{2}\{ & (<b>\partial\partial W + 2<b\partial\partial h^B>V^B)\partial\partial W + <\vartheta>\partial\dot{W}\partial\dot{W} + \\
& + <B\partial\partial h^A \partial\partial h^B>V^A V^B - <\mu>\dot{W}\dot{W} + \\
& + (\underline{<\vartheta\partial h^A \partial h^B>} - \underline{<\mu h^A h^B>})\dot{V}^A\dot{V}^B\} + <p>W,
\end{aligned}
\tag{5}
$$

with underlined terms, which depend on the microstructure parameter l.

4. The outline of the asymptotic modelling

In the asymptotic modelling, cf. [13], [4], the asymptotic procedure is applied. Using the asymptotic decomposition $w_\varepsilon(x,y,t) = U(y,t) + \varepsilon^2 \tilde{h}_\varepsilon^A(x,y)Q^A(y,t)$ in equation (2) and bearing in mind the limit passage $\varepsilon \to 0$ terms $O(\varepsilon)$ are neglected in final equations.

Using the above asymptotic decomposition and averaging operator (3) to the resulting formula, the asymptotic averaged lagrangean $<\Lambda_0>$ is obtained:

$$
\begin{aligned}
<\Lambda_0> = -\tfrac{1}{2}\{ & (<b>\partial\partial W + 2<b\partial\partial h^B>V^B)\partial\partial W + <\vartheta>\partial\dot{W}\partial\dot{W} + \\
& + <B\partial\partial h^A \partial\partial h^B>V^A V^B - <\mu>\dot{W}\dot{W}\} + <p>W.
\end{aligned}
\tag{6}
$$

This model does not describe effects of the microstructure size.

5. Governing equations of presented models

Equations of two models are presented here: *the tolerance model*, *the asymptotic model*.

Substituting $<\Lambda_h>$, (5), to the proper Euler-Lagrange equations, after some manipulations we arrive at the following system of equations for $W(\cdot,t)$ and $V^A(\cdot,t)$:

$$
\begin{aligned}
& \partial\partial(<b>\partial\partial W + <b\partial\partial h^B>V^B) + <\mu>\ddot{W} - <\vartheta>\partial\partial\ddot{W} = <p>, \\
& <b\partial\partial h^A>\partial\partial W = -<B\partial\partial h^A \partial\partial h^B>V^B - (\underline{<\mu h^A h^B>} + \underline{<\vartheta\partial h^A \partial h^B>})\ddot{V}^B.
\end{aligned}
\tag{7}
$$

Equations (7) together with micro-macro decomposition (4) stand *the tolerance model of thin periodic plate bands*. These equations describe free vibrations of these plates and take into account the effect of the microstructure size on them by the underlined terms dependent on the microstructure parameter l. In contrast to equation (2), which has non-continuous, highly oscillating and periodic coefficients, equations (7) have constant coefficients. The basic unknowns W, V^A, $A=1,\ldots,N$, are slowly-varying functions in $x \equiv x_1$. It can be observed that boundary conditions have to be formulated only for *the macrodeflection* W on all edges.

Using the asymptotic modelling procedure, shown in [13, 4], equations of an approximate model, without the effect of the microstructure size, can be obtained in the following form:

$$\begin{aligned}&\partial\partial(\langle b\rangle\partial\partial W+\langle b\partial\partial h^B\rangle V^B)+\langle\mu\rangle\ddot{W}-\langle\vartheta\rangle\partial\partial\ddot{W}=\langle p\rangle,\\&\langle b\partial\partial h^A\rangle\partial\partial W=-\langle B\partial\partial h^A\partial\partial h^B\rangle V^B.\end{aligned}\tag{8}$$

Equations (8) stand the governing equations of *the asymptotic model* of periodic plate bands. It can be observed that these equations can be also derived by neglecting the underlined terms in equations (7). The asymptotic model equations have also constant coefficients, but they describe free vibrations of thin plates under consideration on the macrolevel only.

6. Applications – free vibrations of periodic plate bands with various boundary conditions

Let us consider a thin periodic plate band with span L along the x-axis, neglecting the loading p, p=0. The material properties of this plate are independent of the x_2-coordinate. Let us assume the constant plate thickness d.

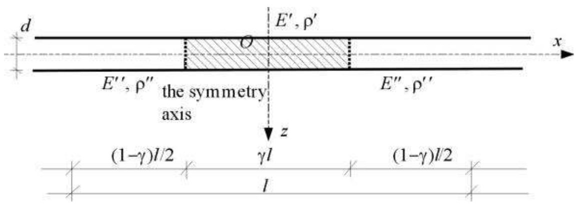

Figure 2. A cell of the plate band

It is assumed that the plate band is made of two different homogeneous isotropic materials, with properties described by Young's moduli E'', E' and mass densities ρ'', ρ':

$$E(y)=\begin{cases}E', & \text{for} \quad y\in((1-\gamma)l/2,(1+\gamma)l/2),\\ E'', & \text{for} \quad y\in[0,(1-\gamma)l/2]\cup[(1+\gamma)l/2,l],\end{cases}\tag{8}$$

$$\rho(y)=\begin{cases}\rho', & \text{for} \quad y\in((1-\gamma)l/2,(1+\gamma)l/2),\\ \rho'', & \text{for} \quad y\in[0,(1-\gamma)l/2]\cup[(1+\gamma)l/2,l],\end{cases}\tag{9}$$

where γ is a distribution parameter of material properties, cf. Figure 2; the Poisson's ratio $\nu\equiv\nu''=\nu'$ is constant.

Our considerations are restricted to only one fluctuation shape function, i.e. $A=N=1$. Denote $h\equiv h^1$, $V\equiv V^1$. Hence, micro-macro decomposition (4) has the form:

$$w(x,t)=W(x,t)+h(x)V(x,t),\tag{10}$$

where the fluctuation shape function $h(x)$ assumed for the cell shown in Figure 2, takes the form:

$$h(y)=l^2[\cos(2\pi y/l)+c], \qquad y\in\Delta(x), \quad x\in\Lambda,\tag{11}$$

with parameter c is a constant determined by $\langle\mu h\rangle=0$:

$$c=\sin(\pi\gamma)(\rho'-\rho'')\{\pi[\rho'\gamma+\rho''(1-\gamma)]\}^{-1}.\tag{12}$$

Under denotations:

$$\breve{B} = \langle b \rangle, \quad \widehat{B} = \langle b\partial\partial h \rangle, \quad \overline{B} = \langle b\partial\partial h\partial\partial h \rangle,$$
$$\breve{\mu} = \langle \mu \rangle, \quad \overline{\mu} = l^{-4} \langle \mu hh \rangle, \quad \breve{\vartheta} = \langle \vartheta \rangle, \quad \overline{\vartheta} = l^{-2} \langle \vartheta\partial h\partial h \rangle, \tag{13}$$

tolerance model equations (7) can be written as:

$$\partial\partial(\breve{B}\partial\partial W + \widehat{B}V) + \breve{\mu}\ddot{W} - \breve{\vartheta}\partial\partial\ddot{W} = 0,$$
$$\widehat{B}\partial\partial W + \overline{B}V + l^2(l^2\overline{\mu} + \overline{\vartheta})\ddot{V} = 0, \tag{14}$$

however, asymptotic model equations (8) take the form of one equation:

$$\partial\partial[(\breve{B} - \widehat{B}^2/\overline{B})\partial\partial W] + \breve{\mu}\ddot{W} - \breve{\vartheta}\partial\partial\ddot{W} = 0. \tag{15}$$

Certain approximate formulas of free vibrations frequencies for periodic plate bands with various boundary conditions can be obtained applying the known Ritz method, cf. [4-6]. Using this method the maximal strain energy Υ_{max} and the maximal kinetic energy K_{max} are determined. For the plate band solutions to equations (14) and (15), which are applied in the Ritz method, can be assumed in the form:

$$W(x,t) = A_W \Xi(\alpha x)\cos(\omega t), \qquad V(x,t) = A_V \Theta(\alpha x)\cos(\omega t), \tag{16}$$

where: α is a wave number; ω is a free vibration frequency; functions $\Xi(\cdot)$ and $\Theta(\cdot)$ are eigenvalue functions for the macrodeflection and the fluctuation amplitude, respectively, which have to satisfy the proper boundary conditions for x=0, L. Denote the first and second order derivatives of functions $\Xi(\cdot)$ and $\Theta(\cdot)$ by:

$$\partial\Xi(\alpha x) \equiv \alpha\widetilde{\Xi}(\alpha x), \quad \partial\Theta(\alpha x) \equiv \alpha\widetilde{\Theta}(\alpha x), \quad \partial\partial\Xi(\alpha x) \equiv \alpha^2\overline{\Xi}(\alpha x), \quad \partial\partial\Theta(\alpha x) \equiv \alpha^2\overline{\Theta}(\alpha x). \tag{17}$$

Moreover, let us introduce denotations:

$$\breve{B} = \tfrac{d^3}{12(1-\nu^2)}[E''(1-\gamma) + \gamma E']\int_0^L [\overline{\Xi}(\alpha x)]^2 dx, \quad \widehat{B} = \tfrac{\pi d^3}{3(1-\nu^2)}(E' - E'')\sin(\pi\gamma)\int_0^L \overline{\Xi}(\alpha x)\Theta(\alpha x)dx,$$
$$\overline{B} = \tfrac{(\pi d)^3}{3(1-\nu^2)}\{(E' - E'')[2\pi\gamma + \sin(2\pi\gamma)] + 2\pi E''\}\int_0^L [\Theta(\alpha x)]^2 dx,$$
$$\breve{\mu} = d[(1-\gamma)\rho'' + \gamma\rho']\int_0^L [\Xi(\alpha x)]^2 dx, \qquad \breve{\vartheta} = \tfrac{d^3}{12}[(1-\gamma)\rho'' + \gamma\rho']\int_0^L [\widetilde{\Xi}(\alpha x)]^2 dx,$$
$$\overline{\mu} = \tfrac{d}{4\pi}\{(\rho' - \rho'')[2\pi\gamma + \sin(2\pi\gamma)] + 2\pi\rho''\}\int_0^L [\Theta(\alpha x)]^2 dx +$$
$$+ \tfrac{d}{\pi}(\rho' - \rho'')c[\pi c\gamma - 2\sin(\pi\gamma)]\int_0^L [\Theta(\alpha x)]^2 dx + d\rho'' c^2 \int_0^L [\Theta(\alpha x)]^2 dx,$$
$$\overline{\vartheta} = \tfrac{\pi d^3}{12}\{(\rho' - \rho'')[2\pi\gamma - \sin(2\pi\gamma)] + 2\pi\rho''\}\int_0^L [\Theta(\alpha x)]^2 dx, \tag{18}$$

Using the conditions of the Ritz method:

$$\tfrac{\partial(\Upsilon_{max} - K_{max})}{\partial A_W} = 0, \quad \tfrac{\partial(\Upsilon_{max} - K_{max})}{\partial A_V} = 0, \tag{19}$$

and make some manipulations we arrive at the following formulas:

$$(\omega_{-,+})^2 \equiv \frac{l^2(l^2\overline{\mu} + \overline{\vartheta})\alpha^4\breve{B} + (\breve{\mu} + \alpha^2\breve{\vartheta})\overline{B}}{2(\breve{\mu} + \alpha^2\breve{\vartheta})l^2(l^2\overline{\mu} + \overline{\vartheta})} \mp$$
$$\mp \frac{\sqrt{[l^2(l^2\overline{\mu} + \overline{\vartheta})\alpha^4\breve{B} - (\breve{\mu} + \alpha^2\breve{\vartheta})\overline{B}]^2 + 4(\alpha^2\widehat{B})^2 l^2(\breve{\mu} + \alpha^2\breve{\vartheta})(l^2\overline{\mu} + \overline{\vartheta})}}{2(\breve{\mu} + \alpha^2\breve{\vartheta})l^2(l^2\overline{\mu} + \overline{\vartheta})}, \tag{20}$$

of *the lower frequency* ω_- *of free macro-vibrations* and *the higher frequency* ω_+ *of free micro-vibrations*, respectively, in the framework of *the tolerance model*.

Calculations can be made for various cases of boundary conditions:

- the simply supported plate band: $\Xi(0) = \partial\partial\Xi(0) = \Xi(L) = \partial\partial\Xi(L) = 0$;
- the plate band clamped on both edges: $\Xi(0) = \partial\Xi(0) = \Xi(L) = \partial\Xi(L) = 0$;

- the clamped-hinged plate band: $\Xi(0)=\partial\Xi(0)=\Xi(L)=\partial\partial\Xi(L)=0$;
- the cantilever plate band: $\Xi(0)=\partial\Xi(0)=\partial\partial\Xi(L)=\partial\partial\partial\Xi(L)=0$.

7. Remarks

In this paper *the tolerance model governing equations of thin periodic plate bands* are presented and applied to analyse free vibrations of them. The tolerance modelling replaces the governing differential equation with non-continuous, periodic coefficients by the system of differential equations with constant coefficients, which involve terms with the microstructure parameter. The tolerance model describes the effect of the microstructure size on vibrations. Hence, there are calculated the lower free vibration frequency and the higher free vibration frequency, which is related to the microstructure, for plate bands with various boundary conditions. These calculations are made using the procedure of the Ritz method.

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