

## **The Dynamics of a Coupled Mechanical System with Spherical Pendulum**

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### **Abstract**

The nonlinear response of a three degree of freedom vibratory system with spherical pendulum in the neighbourhood internal and external resonance is investigated. It was assumed that spherical pendulum is suspended to the main body which is suspended by the element characterized by elasticity and damping and is excited harmonically in the vertical direction. The equation of motion have been solved numerically. In this type system one mode of vibration may excite or damp another one, and for except different kinds of periodic vibrations there may also appear chaotic vibration.

**Keywords:** Spherical pendulum, energy transfer, coupled oscillators, chaos

### **1. Introduction**

The subject of this work is investigation of initial conditions effect on dynamics of a three degree of freedom system with spherical pendulum. Dynamical systems with element of the mathematical or physical pendulum type have important applications. Different kind of coupled autoparametric oscillators with simply pendulums is presented in book [1]. The real pendulum is a spherical character. Spherical pendulum was investigated by a lot of researches. Spherical pendulum subject to parametric excitation was studied by Miles and Zou [2], and with kinematic external excitation by Naprstek and Fischer [3]. The bifurcation behaviour of a spherical pendulum where the suspension point is harmonically excited in both vertical and horizontal directions was presented by Leung and Kung [4], spherical pendulum with moving pivot by Mitrev and Grigorov [5], stochastic analysis of a spring spherical pendulum was done by Viet [6], the dynamics coupled spherical pendulums was studied by Witkowski at all [7].

In the present paper is assumed that the spherical pendulum is suspended to the flexible element, so in this system may occur the autoparametric excitation as a result of inertial coupling.

## 2. System description and equation of motion

The investigated system is shown in Figure 1.

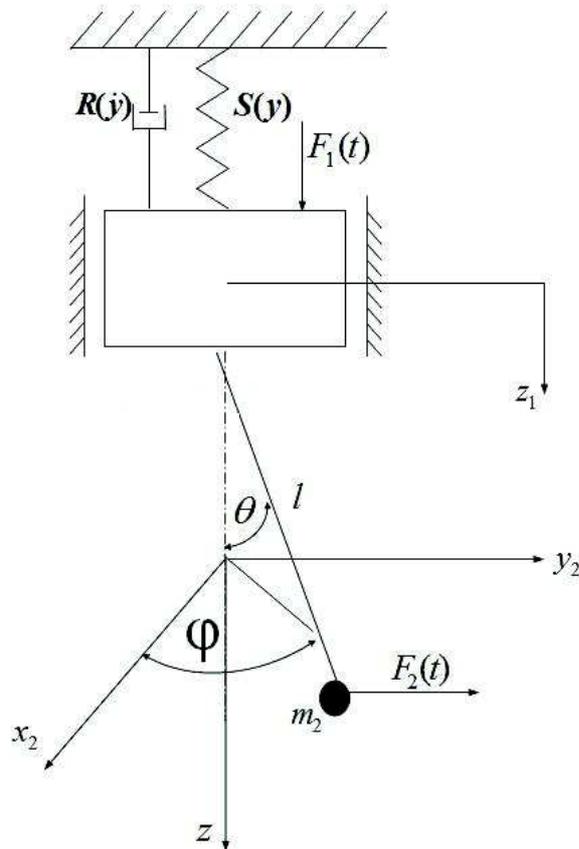


Figure 1. Schematic diagram of system

The system consists of a body of mass  $m_1$  suspended on the flexible element of rigidity  $k$  and damping  $c$  and a spherical pendulum of length  $l$  and mass  $m_2$  suspended on the body of mass  $m_1$ . The body of mass  $m_1$  subjected to harmonic vertical excitation and the spherical pendulum subjected to harmonic horizontal excitation.

The spherical pendulum is similar to the simple pendulum, but moves in 3-dimensional space, so we need to introduce the new variable  $\varphi$  in order to describe the rotation of the pendulum in space  $xy$ . The position of the body of mass  $m_1$  is described by coordinate  $z$  and position of the pendulum is describe by coordinate  $z$  and two angles:  $\Theta$  and  $\varphi$ . Angle  $\Theta$  is the deflections of pendulum measured from the vertical line. This

system has three degrees of freedom . The equations of motion are derived as Lagrange's equations.

The kinetic energy  $E_k$  is the sum of the energy two bodies

$$E_k = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1 \dot{z}^2}{2} + \frac{m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)}{2} \quad (1)$$

where

$$\begin{aligned} x_2 &= l \sin \theta \cos \varphi \\ y_2 &= l \sin \theta \sin \varphi \\ z_2 &= z + l \cos \theta \end{aligned} \quad (2)$$

The kinetic energy  $E_k$  are given by the expression

$$E_k = \frac{1}{2} (m_1 + m_2) \dot{z}^2 + \frac{1}{2} m_2 (l \dot{\theta}^2 + l^2 \dot{\varphi}^2 \sin^2 \theta - 2 l \dot{z} \dot{\theta} \sin \theta) \quad (3)$$

The potential energy  $E_p$  are given by the expression

$$E_p = -(m_1 + m_2)g(z + z_{st}) + m_2 g(l - l \cos \theta) + \frac{k(z + z_{st})^2}{2} \quad (4)$$

Assuming that the exciting forces are in form:  $F(t) = P_1 \cos \nu_1 t$   $F_2(t) = P_2 \cos \nu_2 t$ , the equations of motion of the system are in form

$$\begin{aligned} (m_1 + m_2) \ddot{z} - m_2 l \ddot{\theta} \sin \theta - m_2 l \dot{\theta}^2 \cos \theta + kz + c\dot{z} &= P_1 \cos \nu_1 t \\ m_2 l^2 \ddot{\theta} - m_2 l \ddot{z} \sin \theta - m_2 l^2 \dot{\varphi}^2 \sin \theta \cos \theta + m_2 g l \sin \theta &= l \cos \theta \sin \varphi P_2 \cos \nu_2 t \\ m_2 l^2 \ddot{\varphi} \sin^2 \theta + 2m_2 l^2 \dot{\varphi} \dot{\theta} \sin \theta \cos \theta &= l \sin \theta \cos \theta P_2 \cos \nu_2 t \end{aligned} \quad (5)$$

By introducing the dimensionless time and dimensionless parameters

$$\begin{aligned} \tau = \omega_1 t, \quad \omega_1^2 = \frac{k}{m_1 + m_2}, \quad \omega_2^2 = \frac{g}{l}, \quad \beta = \frac{\omega_2}{\omega_1}, \quad \gamma = \frac{c}{(m_1 + m_2)\omega_1}, \quad \bar{z} = \frac{z}{l} \\ a = \frac{m_2}{m_1 + m_2}, \quad A_1 = \frac{P_1}{(m_1 + m_2)\omega_1^2}, \quad A_2 = \frac{P_2}{m_2 l \omega_1^2}, \quad \mu_1 = \frac{\nu_1}{\omega_1}, \quad \mu_2 = \frac{\nu_2}{\omega_1} \end{aligned} \quad (6)$$

We can transform (5) into dimensionless form

$$\begin{aligned}\ddot{z} - a\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta + z + \gamma\dot{z} &= A_1\cos\mu_1\tau \\ \ddot{\theta} - \ddot{z}\sin\theta - \dot{\varphi}^2\sin\theta\cos\theta + \beta^2\sin\theta &= A_2\cos\theta\sin\varphi\cos\mu_2\tau \\ \ddot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\cos\theta &= A_2\cos\theta\cos\mu_2\tau\end{aligned}\quad (7)$$

(where the overbars denoting nondimensionalisation are omitted for convenience). After transformations equations of motion can be written in form easier to calculations

$$\begin{aligned}\ddot{z} &= [(A_1\cos\mu_1\tau + a\dot{\theta}^2\cos\theta - z - \gamma\dot{z}) + a(\varphi^2\sin\theta\cos\theta - \beta^2\sin\theta)]\sin\theta/(1 - a\sin^2\theta) \\ \ddot{\theta} &= \dot{\varphi}_2\sin\theta\cos\theta - \beta^2\sin\theta + [(A_1\cos\mu_1\tau + a\dot{\theta}^2\cos\theta - z - \gamma\dot{z}) + a(\dot{\varphi}_2\sin\theta\cos\theta + \\ &\quad - \beta^2\sin\theta)\sin^2\theta]\sin\theta/(1 - a\sin^2\theta) \\ \ddot{\varphi} &= (A_2\cos\theta\cos\mu_2\tau - 2\dot{\varphi}\dot{\theta}\cos\theta)/\sin\theta\end{aligned}\quad (8)$$

### 3. Numerical results

Equations (8) are solved numerically by using R-K method with step length variable. The calculations are carried out for different values of parameters of the system and for different initial conditions. Exemplary time histories of displacements  $z$  and  $\theta$  obtained for the initial conditions for the body of mass  $m_1$  are presented in Figure 2, where we can observe the energy transfer between the modes of vibration in a closed cycle. In this case spherical pendulum behaviour is the same than simple pendulum and the motion of pendulum is in vertical plane (angle  $\varphi$  is constant). The diagram of internal resonance for initial conditions put on the displacements is presented in Figure 4 and it is similar to simple pendulum presented in work [1]. We observe resonance excitation for frequency ratio  $\beta=0.5$ . In this case assuming the simple pendulum results are good.

When the initial conditions are put on the displacements and on the velocities ( $z(0) = 0; \dot{z}(0) = 0; \theta(0) = 5^\circ; \dot{\theta}(0) = -0.04; \varphi(0) = 0; \dot{\varphi}(0) = -0.96$ ) we observe influence of angle  $\varphi$ . (Figures 3). Exemplary internal resonance in this case we observe for frequency ratio  $\beta=0.51$  (Figure 5).

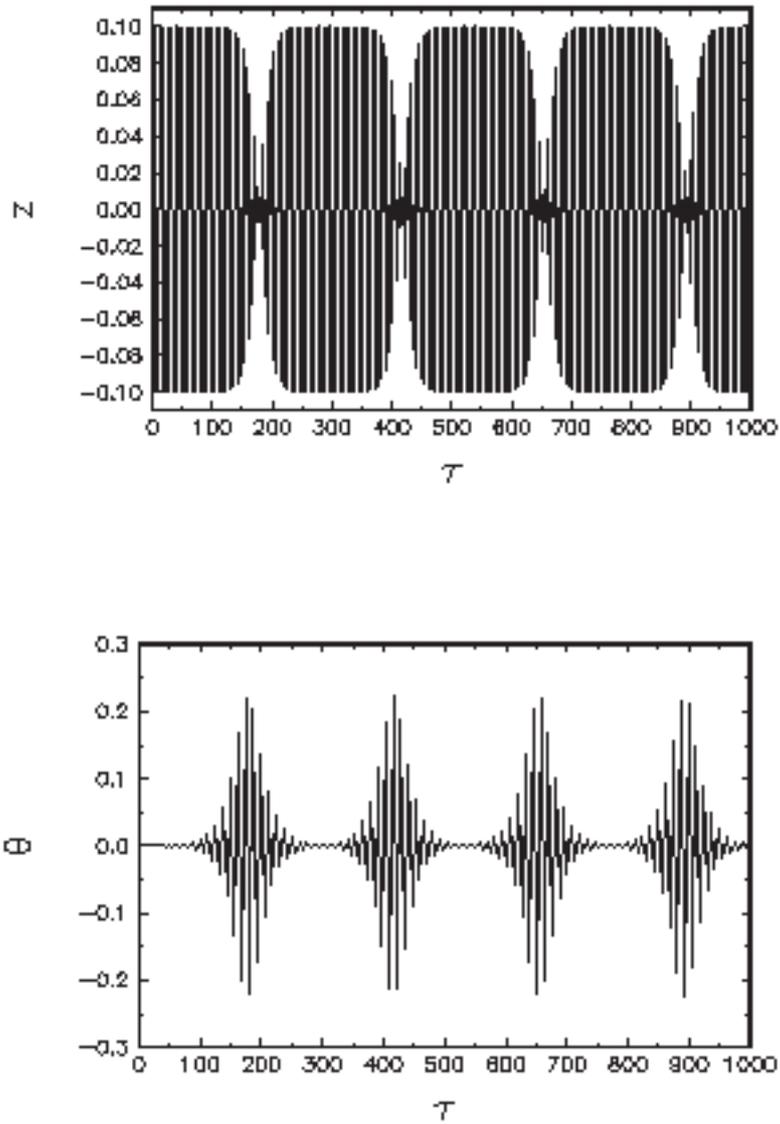


Figure 2. Time history for:  $a=0.8$ ;  $\beta=0.5$ ;  $\gamma=0$ ;  $A_1=A_2=0$ ;  
 $z(0)=0.1$ ;  $\Theta(0)=0.005^\circ$ ;  $\varphi(0)=0$

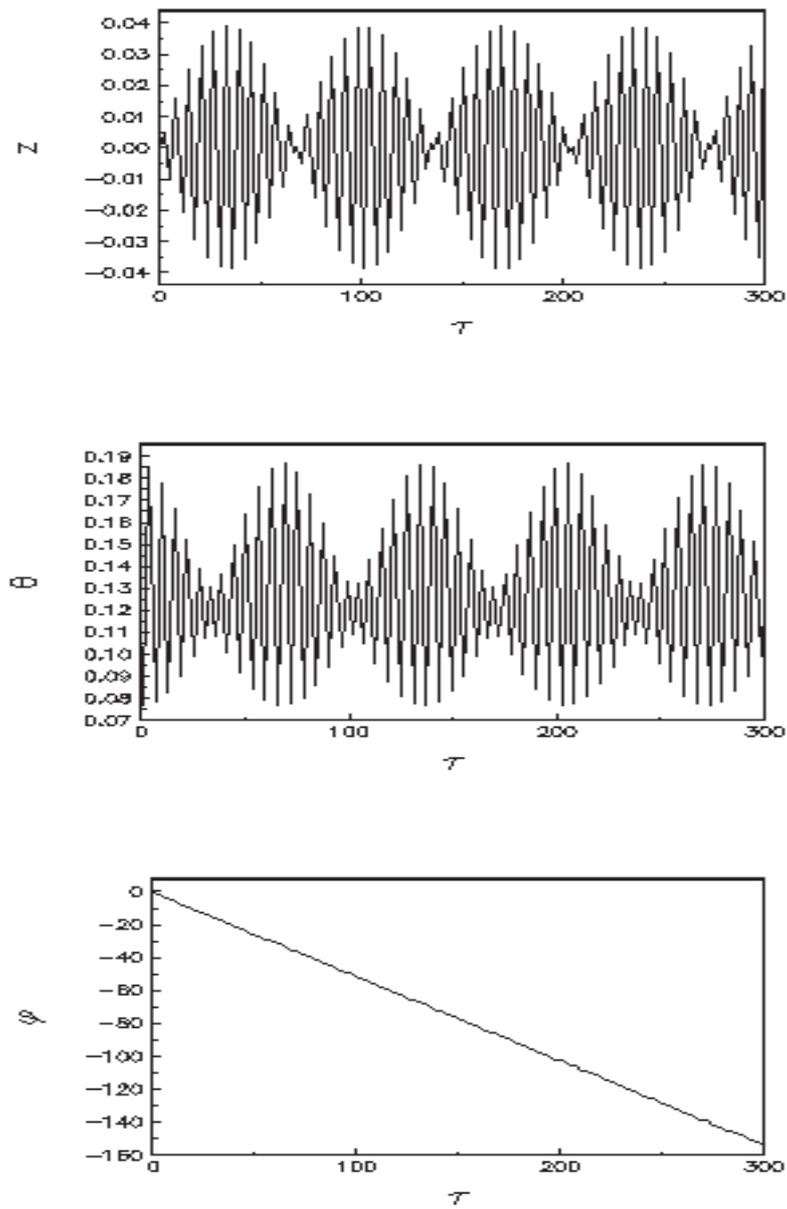


Figure 3. Time history for:  $a=0.5$ ;  $\beta=0.51$ ;  $\gamma=0$ ;  $A_1=A_2=0$ ;  
 $z(0)=0$ ;  $\dot{z}(0)=0$ ;  $\theta(0)=5^\circ$ ;  $\dot{\theta}(0)=-0.04$ ;  $\varphi(0)=0$ ;  $\dot{\varphi}(0)=-0.96$

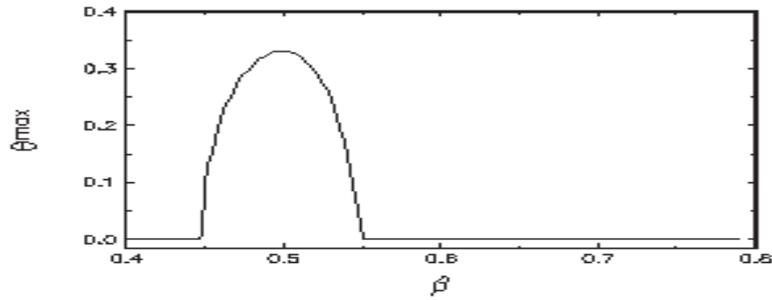


Figure 4. Internal resonance for:  $a=0.8; \gamma=0; A_1=A_2=0;$   
 $z(0)=0.1; \Theta(0)=0.005^\circ; \varphi(0)=0$

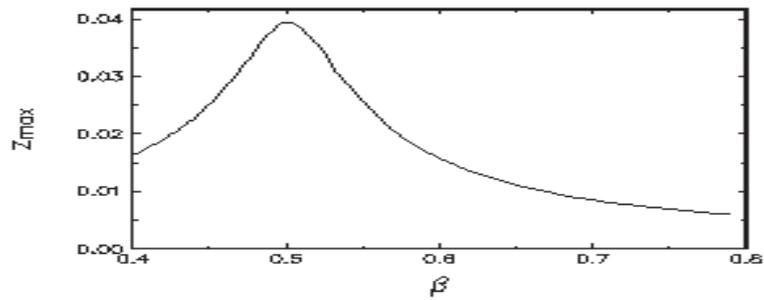


Figure 5. Internal resonance for:  $a=0.5; \gamma=0; A_1=A_2=0;$   
 $z(0)=0; \dot{z}(0)=0; \theta(0)=5^\circ, \dot{\theta}(0)=-0.04; \varphi(0)=0; \dot{\varphi}(0)=-0.96$

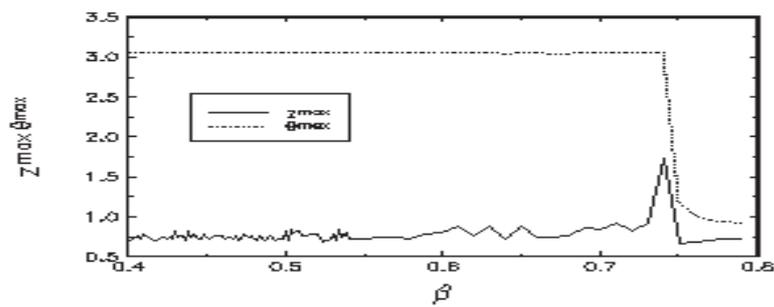


Figure 6. Internal resonance for:  $a=0.2; \gamma=0; A_1=A_2=0;$   
 $z(0)=0; \dot{z}(0)=0.65; \theta(0)=50^\circ, \dot{\theta}(0)=-0.04; \varphi(0)=0; \dot{\varphi}(0)=-0.296$

But when the initial conditions are put on the displacements and on the velocities ( $z(0) = 0; \dot{z}(0) = 0.65; \theta(0) = 50^\circ; \dot{\theta}(0) = -0.04; \varphi(0) = 0; \dot{\varphi}(0) = -0.296$ ) we observe influence of angle  $\varphi$  and internal resonance area in this case we observe for frequency ratio near  $\beta=0.75$  (Figure 6). In this case  $\varphi$  described the rotation of the pendulum around axis  $z$ , so assuming the spherical pendulum we have the results more similar to the real system.

### 3. Conclusions

The influence of initial conditions on the behaviour of an autoparametric system with spherical pendulum is very interesting, because sometimes when initial conditions are put on the displacements spherical pendulum is similar to simple pendulum (angle  $\varphi$  is const.), but when the initial conditions are put on the velocities we observe influence of angle  $\varphi$ . It is important, because near internal and external resonance area can existence the different motion - regular or chaotic. The autoparametric systems are very sensitive on nonlinearities. The spherical pendulum is more similar to the real systems then the simply pendulum.

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