Effect of Temperature on Vibrations of Laminated Layer

Ewelina PAZERA
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, Poland,
ewelina.pazera@p.lodz.pl

Jarosław JĘDRYSIAK
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, Poland,
jaroslaw.jedrysiak@p.lodz.pl

Abstract

In this note the influence of temperature on vibrations of laminated layer made of two different materials is presented. The macroscopic properties of this layer are changing continuously along one direction x_1, perpendicular to the laminas. To obtain the equations describing this problem, the tolerance averaging technique is used [1]. In this work, three models are proposed: the tolerance and the asymptotic-tolerance model, taking into account the effect of the microstructure size on the overall behaviour of this type of structures, and the asymptotic model, which equations omit this effect. To solve the equations of these three models the finite difference method is used

Keywords: thermoelasticity, vibrations, non-periodic laminates, tolerance averaging technique

1. Introduction

The objects under consideration are laminated layers, made of two different materials, non-periodically distributed as microlaminas along a direction perpendicular to the laminas. The macroscopic properties of these structures are changing continuously along direction x_1 (normal to the laminas). This type of structures can be called the functionally graded laminates [2], and a microstructure can be realised as a uniform (λ = const) or non-uniform ($\lambda = \lambda(x)$) distribution of the cells. In this note the uniform distribution of the cells is considered, so the thickness of the cells is constant and denoted by λ, as shown in the Fig. 1. The basic cell in reference to these laminated layers cannot be defined in a simply way, and thermoelasticity issues can be considered in relation to micromechanical models with idealized geometry. To analyse the various problems related to the layers with functional gradation of properties, which are not homogeneous in microscale, the assumptions of idealization similar to these used to analyse periodic composites, can be applied. Between the methods, which are used for periodic structures, the asymptotic homogenization and the homogenization based on the microlocal parameters, should be mentioned [2, 3]. These methods can be modified and adopted to describe the structures with functional gradation of properties.

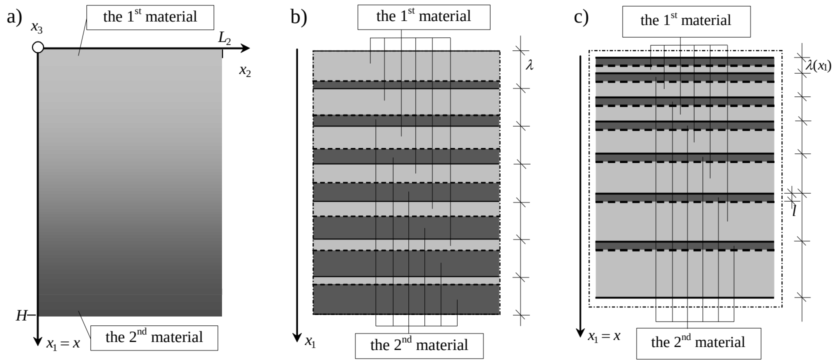

Figure 1. A cross-section of the laminate: a) the macro-level, b) the micro-scale with uniform distribution of cells, c) the micro-scale with non-uniform distribution of cells

In the analysis of various problems related to the layered structures also the alternative methods can be used, for example the higher order theory to analyse the overall behaviour of functionally graded structures, the finite element method in the analysis of sandwich beams, layered or sandwich plates, strong form collocation method for solving laminated composite plates, generalized differential quadrature in the analysis of laminated doubly-curved shells or differential quadrature finite element method in the analysis of composite plates [4-12].

Unfortunately, most of the known methods do not take into account the impact of the microstructure size in model equations of the functionally graded composites.

In this work, to obtain the governing equations, which give the possibility to consider this influence, the tolerance averaging technique is used [13, 14]. By using this technique the various thermomechanical issues of periodic structures were considered [15, 16]. Moreover the tolerance modelling was used to investigate different thermal problems related to the functionally graded media [17-21]. Additionally this way of modelling was used to describe thermal issues in a two-phase hollow cylinder, vibrations of layered plates or dynamic problems for thin microstructured transversally graded and cylindrical shells [22-28].

The basic aim of this work is to obtain and present the equations of the tolerance, asymptotic-tolerance and asymptotic model describing the influence of the temperature on vibrations of laminated layer. The equations of two of them involve terms, which describe the effect of the microstructure size.

2. Modelling foundations

The considered issue can be described by the known following equations:

$$\begin{aligned} &\partial_j\left(C_{ijkl}\partial_l u_k\right)-\rho\ddot{u}_i=\partial b_{ij}\theta+b_{ij}\partial_j\theta,\\ &\partial_j\left(k_{ij}\partial_i\theta\right)=c\dot{\theta}+T_0 b_{ij}\partial_j\dot{u}_i, \end{aligned} \tag{1}$$

where i, j, k, l accept values 1, 2, 3, by u_i and θ the unknown displacements along the x_i-axis and the temperature are denoted and the material coefficients (tensor of elasticity C_{ijkl}, tensor of heat conduction k_{ij}, tensor of thermal modules b_{ij}, mass density ρ, specific heat c) are non-continuous and highly-oscillating material coefficients.

The basic concepts related to the tolerance averaging technique are averaging operation, tolerance-periodic, slowly-varying and highly-oscillating functions.

The averaging operator is defined by the following equation:

$$\left\langle \partial^i f \right\rangle(x) \equiv |\Delta|^{-1} \int_{\Delta(x)} \tilde{f}^{\,i}(x,z)dz, \tag{2}$$

where i accept values 0, 1, 2, $x\epsilon\Omega$, Ω is a space limited area in R, $z\epsilon\Delta(x)$, $\Delta(x) = x + \Delta$ is a cell with the centre in $x\epsilon$R, $\Delta\equiv[-\lambda/2,\ \lambda/2]$ is the basic cell and to mark a periodic approximation of the gradient $\partial^i f$ in $\Delta(x)$, a sign $\sim$ is introduced. By $\Omega\times\Xi$ the space limited area in R^3 is denoted, where Ω is included in R and Ξ is included in R^2, then the coordinates in Ω are denoted by $x = x_1$ or $z = z_1$ and coordinates in Ξ are denoted by $\varsigma = (\varsigma_1, \varsigma_2)$.

By f the tolerance-periodic function in reference to the basic cell Δ and tolerance parameter δ is denoted. The function can be called the tolerance-periodic function, when the succeeding terms are fulfilled:

$$\begin{gathered}(\forall x \in \Omega)\left(\exists \tilde{f}^{(i)}(x,\cdot) \in \mathrm{H}^0(\Delta)\right)\left(\left\|\partial^i f\,|_{\Omega x}(\cdot) - \tilde{f}^{(i)}(x,\cdot)\right\|_{\mathrm{H}^0(\Omega_x)} \le \delta\right),\\ \int\limits_{\Delta(\cdot)} \tilde{f}^{(i)}(\cdot,z)dz \in C^0(\bar{\Omega}),\end{gathered} \tag{3}$$

where i accept values 0, 1, 2 and $H^0(\Delta)$ is a space of Δ-periodic functions, which can be square integrable.

The function u can be called the slowly-varying function in reference to the basic cell Δ and tolerance parameter δ, when the function u is a tolerance-periodic function and the following term is executed:

$$(\forall x \in \Omega)\left(\tilde{u}^{(i)}(x,\cdot)\Big|_{\Delta(x)} = \partial^i u(x)\right), \tag{4}$$

where i accept values 0, 1, 2 and a periodic approximation of $\partial^i u(\cdot)$ is a constant function in an area of $\Delta(x)$ for every $x\epsilon\Omega$.

By h the highly-oscillating function in reference to the basic cell Δ and tolerance parameter δ is denoted. The function can be called the highly-oscillating if this function is a tolerance periodic function and the following term is fulfilled:

$$(\forall x \in \Omega)\left(\tilde{h}^{(i)}(x,\cdot)\Big|_{\Delta(x)} = \partial^i \tilde{h}(x)\right), \tag{5}$$

where i accept values 0, 1, 2.

3. Modelling procedures

The tolerance averaging technique based on two main assumptions. The first assumption is the micro-macro decomposition, where the basic unknowns can be taken as a sum of the averaged part and the oscillating part, which can be expressed as a product of a known fluctuation shape function and fluctuation amplitude called the new basic unknown, according to the following equations:

$$\begin{aligned} u_i(x,\varsigma,t) &= w_i(x,\varsigma,t) + h^A(x)v_{iA}(x,\varsigma,t), \\ \theta(x,\varsigma,t) &= \vartheta(x,\varsigma,t) + g^B(x)\psi_B(x,\varsigma,t), \end{aligned} \tag{6}$$

where w_i (i = 1, 2, 3) and ϑ are called the microdisplacements and the macrotemperature, respectively, fluctuation amplitudes of displacements and temperature are denoted by v_{iA} and ψ_B, respectively and h^A and g^B are known fluctuation shape functions and have to be defined for each analysed case. In this note one fluctuation shape function is considered ($A = B = 1$), expressed by the succeeding equation:

$$h(x,z) = g(x,z) = \begin{cases} -\lambda\sqrt{3}\dfrac{1}{v_1(x)}\left(\dfrac{2z}{\lambda} + v_2(x)\right) & \text{for } z \in \left(-\dfrac{\lambda}{2}, -\dfrac{\lambda}{2} + v_1(x)\lambda\right) \\ \lambda\sqrt{3}\dfrac{1}{v_2(x)}\left(\dfrac{2z}{\lambda} - v_1(x)\right) & \text{for } z \in \left(-\dfrac{\lambda}{2} + v_1(x)\lambda, \dfrac{\lambda}{2}\right) \end{cases}, \tag{7}$$

where $v_1(x)$ and $v_2(x)$ define the share of the first and the second material in the cell. The proposed fluctuation shape function is a saw-like function and guarantees the continuity of the displacements and the temperature between the layers and between the sublayers.

The second assumption of the tolerance modelling is the periodic approximation of k^{th} derivatives of considered functions, which can be expressed using the following equations:

$$\begin{aligned} \tilde{u}_i^{(k)}(x,z,\varsigma) &= \nabla^k w_i(x,\varsigma) + \partial^k \tilde{h}^A(x,z)v_{iA}(x,\varsigma) + \tilde{h}^A(x,z)\bar{\nabla}^k v_{iA}(x,\varsigma), \\ \tilde{\theta}^{(k)}(x,z,\varsigma) &= \nabla^k \vartheta(x,\varsigma) + \partial^k \tilde{g}^B(x,z)\psi_B(x,\varsigma) + \tilde{g}^B(x,z)\bar{\nabla}^k \psi_B(x,\varsigma). \end{aligned} \tag{8}$$

where $z\epsilon\Delta(x)$, $x\epsilon\Omega$.

To obtain the equations of the proposed models, the orthogonalization method is used, where approximated functions are expanding in series relative to the linear-independent basic functions, according to the following equations:

$$u_i \equiv h^0 w_i + h^A v_{iA}, \;\; \theta \equiv g^0\vartheta + g^B\psi_B, \tag{9}$$

where $h^0 = g^0 = 1$, $A = B = 1$ (one fluctuation shape function is considered) and macrodisplacements, macrotemperature and fluctuation amplitudes are treated as unknowns parameters. Then the residuum functions for the displacements and the temperature were formulated in the form of following equations:

$$\Re_u = \partial_j\left(C_{ijkl}\partial_l u_k\right) - \rho\ddot{u}_i - \partial_j\left(b_{ij}\theta\right), \;\; \Re_\theta = \partial_i\left(k_{ij}\partial_j\theta\right) - c\dot{\theta}, \tag{10}$$

and the terms which have to be fulfilled by this functions, according to the succeeding equations:

$$\left\langle \Re_u \right\rangle(\varsigma) = 0, \;\; \left\langle \tilde{h}^A \Re \right\rangle(\varsigma) = 0, \;\; \left\langle \Re_\theta \right\rangle(\varsigma) = 0, \;\; \left\langle \tilde{g}^B \Re \right\rangle(\varsigma) = 0. \tag{11}$$

4. The model equations

By using the main equations of thermoelasticity, the orthogonalization method, the micro-macro decomposition assumption and a few additional assumptions of the tolerance averaging technique, by doing some manipulations and including the asymmetrical character of the fluctuation shape function, the equations of the tolerance model are obtained in the following form:

$$\begin{aligned}
&\partial_j\left(<C_{ijkl}>\partial_l w_k+<C_{ijk1}\partial h>v_k-<b_{ij}>\vartheta\right)=<\rho>\ddot{w}_i,\\
&\partial_i\left(<k_{ij}>\partial_j\vartheta+<k_{i1}\partial g>\psi\right)-<c\rho>\dot{\vartheta}=\underline{\underline{<T_0 b_{ij}>\partial_j\dot{w}_i}}+\underline{\underline{<T_0 b_{i1}\partial h>\dot{v}_i}},\\
&\partial_\beta\left(\underline{<C_{i\beta k\delta}hh>\partial_\delta v_k}-\underline{<b_{i\beta}gh>\psi}\right)-<C_{i1kl}\partial h>\partial_l w_k=\\
&\qquad=\left(<C_{i1k1}\partial h\partial h>v_k-<b_{i1}\partial h>\vartheta\right)+\underline{<\rho hh>\ddot{v}_i},\\
&\partial_\alpha\left(\underline{<k_{\alpha\beta}gg>\partial_\beta\psi}\right)-<k_{1j}\partial g>\partial_j\vartheta-<k_{11}\partial g\partial g>\psi=\\
&\qquad=\underline{<c\rho gg>\dot{\psi}}+\underline{\underline{<T_0 b_{i\delta}gh>\partial_\delta\dot{v}_i}},
\end{aligned}\tag{12}$$

where the underlined terms are dependent on the microstructure parameter λ and the double-underlined terms, responsible for the full connection between the displacements and the temperature, are treated as negligibly small.

Directly from the equations of the tolerance model, by omitting the underlined terms (by limit passage with λ to zero), the equations of the asymptotic model can be obtained in the following form:

$$\begin{aligned}
&\partial_j\left(<C_{ijkl}>\partial_l w_k+<C_{ijk1}\partial h>v_k-<b_{ij}>\vartheta\right)=<\rho>\ddot{w}_i,\\
&\partial_i\left(<k_{ij}>\partial_j\vartheta+<k_{i1}\partial g>\psi\right)-<c\rho>\dot{\vartheta}=\underline{\underline{<T_0 b_{ij}>\partial_j\dot{w}_i}}+\underline{\underline{<T_0 b_{i1}\partial h>\dot{v}_i}},\\
&-<C_{i1kl}\partial h>\partial_l w_k-\left(<C_{i1k1}\partial h\partial h>v_k-<b_{i1}\partial h>\vartheta\right)=0,\\
&<k_{1j}\partial g>\partial_j\vartheta+<k_{11}\partial g\partial g>\psi=0,
\end{aligned}\tag{13}$$

The equations of the asymptotic-tolerance model can be obtained in two steps. In the first step the solution of the asymptotic model is obtained:

$$\begin{aligned}
u_{0i}\left(x,\varsigma,t\right)&=w_i\left(x,\varsigma,t\right)+h^A(x)v_{iA}\left(x,\varsigma,t\right),\\
\theta_0\left(x,\varsigma,t\right)&=\vartheta\left(x,\varsigma,t\right)+g^B(x)\psi_B\left(x,\varsigma,t\right),
\end{aligned}\tag{14}$$

then in the second step the additional micro-macro decomposition in the form of following equations is used for the main equations of thermoelasticity problems:

$$\begin{aligned}
u_i\left(x,\varsigma,t\right)&=u_{0i}\left(x,\varsigma,t\right)+f^A(x)r_{iA}\left(x,\varsigma,t\right),\\
\theta\left(x,\varsigma,t\right)&=\theta_0\left(x,\varsigma,t\right)+d^B(x)\chi_B\left(x,\varsigma,t\right),
\end{aligned}\tag{15}$$

where the macrodisplacements w_i, the macrotemperature ϑ and the fluctuation amplitudes of the displacements v_i and the temperature ψ are assumed as known functions, r_{iA} and χ_B are the additional fluctuation amplitudes of the displacements and the temperature, respectively, f^A and d^B are new known fluctuation shape functions. Then the additional equations were obtained:

$$
\begin{aligned}
&\underline{<dgk_{\alpha\delta}>\partial_\alpha\partial_\delta\psi}+\underline{<ddk_{\alpha\delta}>\partial_\alpha\partial_\delta\chi}-<\partial dk_{1j}>\partial_j\vartheta-<\partial d\partial gk_{11}>\psi\\
&\qquad\qquad-<\partial d\partial dk_{11}>\chi-\underline{<c\rho dg>\dot{\psi}}-\underline{<c\rho dd>\dot{\chi}}=0,\\
&\partial_\alpha\left(\underline{<C_{i\alpha k\delta}fh>\partial_\delta v_k}+\underline{<C_{i\alpha k\delta}ff>\partial_\delta r_k}\right)-<C_{i1kl}\partial f>\partial_l w_k\\
&\qquad-<C_{i1k1}\partial f\partial h>v_k-<C_{i1k1}\partial f\partial f>r_k-\underline{<\rho hh>\ddot{v}_i}\\
&-\underline{<\rho hf>\ddot{r}_i}-\partial_\alpha\left(\underline{<b_{i\alpha}fg>\psi}+\underline{<b_{i\alpha}fd>\chi}\right)+<b_{i1}\partial f>\vartheta=0,
\end{aligned}
\tag{16}
$$

which with the equations of the asymptotic model are treated as the asymptotic-model equations, where $f = f^A$, $d = d^B$.

The equations of these three models can be used in the analysis of some specific cases, where the distribution of the ingredients is functional, but non-periodic. The presented models can be used e.g. to consider the influence of the temperature on vibrations of laminated layer. It is possible to analyse the forced vibrations of this type of layer, caused by mechanical load (the stresses on the edge of this layer) only, and simultaneously by mechanical load and the temperature.

The equations of presented models can be solved by using e.g. the finite difference method. In this method the solution is substituted by the set of functions values in nodes of discretized area and the derivatives of functions in nodes with appropriate coordinates are replaced by the differential quotients, defined in these nodes. The section along direction parallel to the laminas is constant, usually, and along direction perpendicular to the laminas the nodes have to be defined in the middle of every sublayer, between the sublayers and between the cells. In each of presented models the set of non-homogeneous discretized equations is obtained in the following form:

$$
(1-\alpha)\mathbf{K}\mathbf{q}^{\mathrm{k}+1}+\alpha\mathbf{K}\mathbf{q}^{\mathrm{k}}+\mathbf{K}_t\mathbf{q}^{\mathrm{k}+1}+\alpha\mathbf{K}_t\mathbf{q}^{\mathrm{k}}+(1-\alpha)\mathbf{Q}^{\mathrm{k}+1}+\alpha\mathbf{Q}^{\mathrm{k}}=0,
\tag{17}
$$

where **K** is a matrix of coefficients independent of time coordinate in individual equations, $\mathbf{K}_t$ is a matrix of coefficients dependent on time coordinate in individual equations, **Q** is a vector of free terms, **q** is a vector of unknowns order alternately at individual points and α is a parameter determining the approach in the context of numerical methods. The recommended method in similar issues is the Cranck-Nicholson method, where α is equal to a half.

5. Remarks

The tolerance modelling is a technique, which gives a possibility to replace the system of differential equations with non-continuous, tolerance-periodic and highly-oscillating coefficients, by the differential equations where coefficients are slowly-varying. These equations describing the behaviour of the laminated layers with functional gradation of properties, made of two different materials non-periodically distributed as microlaminas along one direction (normal to the laminas).

One of the most important advantages of using the tolerance averaging technique it is a possibility to analyse the whole structure, without necessity of analysis the problem in a single cell.

By using the tolerance averaging technique the equations of three models are obtained, the tolerance, the asymptotic-tolerance and the asymptotic model. Two of them (the tolerance and the asymptotic-tolerance model) make it possible to analyse the impact of the microstructure size in thermoelasticity issues and the equations of these models describe macro- and microvibrations of considered structures.

In the asymptotic model the influence of the microstructure size is omitted and by using the equations of this model only the macrovibrations can be analysed.

The stage of further research is an analysis of the problem, described by equations with the full connection between the displacements and the temperature.

References

1. J. Jędrysiak, *Thermomechanics of laminates, plates and shells with functionally graded structure*, Publishing House of Łódź University of Technology, Łódź 2010 [in Polish].
2. V. V. Jikov, C. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer Verlag, Berlin-Heidelberg 1994.
3. S.J. Matysiak, *Application of the method of microlocal parameters to problems of periodic thermoelastic composites*, Materials Science, **35**(4) (1999) 521 – 526.
4. J. Aboudi, M.J. Pindera, S.M. Arnold, *Higher-order theory for functionally graded materials*, Composites: Part B, **30** (1999) 777 – 832.
5. J. Aboudi, M. J. Pindera, S. M. Arnold, *Thermo-inelastic response of functionally graded composites*, International Journal of Solids and Structures, **32**(12) (1995) 1675 – 1710.
6. M. Grygorowicz, K. Magnucki, M. Malinowski, *Elastic buckling of a sandwich beam with variable mechanical properties of the core*, Thin-Walled Structures, **87** (2015) 127 – 132.
7. D. Pawlus, *Stability of three-layered annular plate with composite facings*, Applied Composite Materials, **24**(1) (2017) 141 – 158.
8. D. Pawlus, *Dynamic response control of three-layered annular plate due to various parametres of electrorheological core*, Archive of Mechanical Engineering, **63**(1) (2016) 73 – 91.
9. E. Magnucka-Blandzi, Z. Walczak, P. Jasion, L. Wittenbeck, *Modelling of multi-layered band plates with trapezoidal corrugated cores: stability analysis*, Archive of Applied Mechanics, **87**(2) (2017) 219 – 229.
10. N. Fantuzzi, F. Tornabene, *Strong Formulation Isogeometric Analysis (SFIGA) for laminated composite arbitrarily shaped plates*. Composites Part B: Engineering, **96** (2016) 173 – 203.
11. F. Tornebene, N. Fantuzzi, M. Bacciocchi, *On the mechanics of laminated doubly-curved shells subjected to point and line loads*, International Journal of Engineering Science, **109** (2016) 115 – 164.

12. B. Liu, A. J. M. Ferreira, Y. F. Xing, A. M. A. Neves, *Analysis of composite plates using a layerwise theory and a differential quadrature finite element method*, Composite Structures, **156** (2016) 393 – 398.
13. C. Woźniak, E. Wierzbicki, *Averaging techniques in thermomechanics of composite solids*, Publishing House of Częstochowa University of Technology, Częstochowa 2000.
14. C. Woźniak, B. Michalak, J. Jędrysiak, *Thermomechanics of heterogeneous solids and structures,* Publishing House of Łódź University of Technology, Łódź 2008.
15. J. Ignaczak, *A spatial decay estimate for transient thermoelastic process in a composite semispace*, Journal of Thermal Stresses, **23** (2000) 1 – 14.
16. Z. F. Baczyński, *Dynamic thermoelastic processes in microperiodic composites*, Journal of Thermal Stresses, **26** (2003) 55 – 66.
17. J. Jędrysiak, *On the tolerance modelling of thermoelasticity problems for transversally graded laminates*, Archives of Civil and Mechanical Engineering, **11** (2011) 61 – 74.
18. J. Jędrysiak, E. Pazera, *Free Vibrations of Thin Microstructured Plates*, Vibrations in Physical Systems, **26** (2014) 93 – 98.
19. J. Jędrysiak, E. Pazera, *Vibrations of Non-Periodic Thermoelastic Laminates*, Vibrations in Physical Systems, **27** (2016) 175-180.
20. E. Pazera, J. Jędrysiak, *Thermoelastic phenomena in transversally graded laminates*, Composite Structures, **134** (2015) 663 – 671.
21. E. Pazera, J. Jędrysiak, *Effect of microstructure in thermoelasticity problems of functionally graded laminates,* Composite Structures, (2018) https://doi.org/10.1016/j.compstruct.2018.01.082.
22. P. Ostrowski, B. Michalak, *The combined asymptotic-tolerance model of heat conduction in a skeletal micro-heterogeneous hollow cylinder,* Composite Structures, **134** (2015) 343 – 352.
23. J. Marczak, J. Jędrysiak, *Tolerance modelling of vibrations of periodic three-layered plates with inert core*, Composite Structures, **134** (2015) 854 – 861.
24. J. Marczak, J. Jędrysiak, *Vibrations of plate strips with internal periodic structure*, Engineering Transactions, **63**(1) (2015) 93 – 108.
25. J. Marczak, J. Jędrysiak, *Vibrations of periodic sandwich plates with inert core*, Vibrations in Physical Systems, **27** (2016) 265 – 272.
26. B. Tomczyk, P. Szczerba, *Tolerance and asymptotic modelling of dynamic problems for thin microstructured transversally graded shells*, Composite Structures, **162** (2017) 365 – 373.
27. B. Tomczyk, *Dynamic stability of micro-periodic cylindrical shells*, Mechanics and Mechanical Engineering, **14**(2) (2010) 137 – 150.
28. B. Tomczyk, *On the modelling of dynamic problems for biperiodically stiffened cylindrical shells*, Civil and Environmental Engineering Reports, **5** (2010) 179 – 204.