Free Vibrations of the Column Taking Into Account Compressive and Thermal Load

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Abstract

Free vibrations of slender systems are the subject of many scientific and research works. In this work, the boundary problem of free vibrations of a compressed column, which is additionally heat loaded, is considered. The issue of heat flow in the column is solved using the Finite Element Method. Averaged distribution of material properties is obtained in individual segments of the column in subsequent heating times. The mathematical model of free vibrations takes into account the thermal expansion of the material and the effect of changing the Young's modulus resulting from the effect of heat load. The boundary problem of the free vibrations of the considered system is limited to the linear range (the linear component of natural frequency is considered). The influence of the heat source exposure time on the course of characteristic curves (on the plane: load – natural frequency) is determined. The results are presented for various column diameters.

Keywords: free vibrations, Euler's load, heat source, column

1. Introduction

In the case where mechanical constructions may be exposed to vibrations, an important element is to conduct vibration analysis to determine the natural frequencies. Knowing the values of the eigenfrequencies, it is possible to protect the systems against the occurrence of a dangerous resonance phenomenon. For this reason, there are many publications in the scientific literature concerning the problems of natural vibrations of mechanical systems [1-8].

The mathematical model of non-uniform beam vibration has been presented in [1]. The calculations assume variable mass, material damping and bending stiffness along the system length. The system vibration characteristics have been determined. Publication [2] presents a discrete model of non-linear vibrations of a damaged beam, additionally resting on a Winkler elastic foundation. The effect of location and crack size on the value of natural vibration frequency is investigated. The results obtained on the basis of this formulation are compared with the results for a continuous system. Work [3] covers the problem of forced vibration of a cantilever beam. Boundary problem is formulated on the basis of which frequencies and forms of vibrations are determined. The frame vibration analysis is undertaken in [3]. The wave propagation method is used and compared to the FEM .The procedure is extended for interconnecting members at an arbitrary angle. The effects of crack depth and crack location on the in-plane free vibration of cracked frame structures have been investigated numerically by using the Finite Element Method in work [4]. The effects of crack depth and location on the natural frequency of multi-bay and multi-store frame structures are presented in 3D graphs. The results are compared with the results from ANSYS software. Publication [5] relates to non-linear vibrations of imperfect columns. The study shows that the frequency of a loaded system close to the critical value is infinitesimal value. Numerical tests have been confirmed by an experiment. The formulation of the boundary problem of column vibration using Timoshenko's theory with various mounting types is contained in [6]. The characteristic curves are presented and compared with the results obtained on the basis of Bernoulli-Euler theory. Thermal bucking and natural vibration of the beam with a specific boundary condition is analysed in the work [7]. Structure is subject to a uniformly distributed heating and has a frictional sliding end within a clearance. The impact of temperature on the material properties are assumed for high temperatures. The analytical solutions are derived for the buckling temperature and vibration frequency and different parameters of the system are under consideration. In the work [8] the primary resonance of heated beam under specific boundary conditions is analysed. Results from the analytical and numerical approach are compared, taking into account different parameters of the system. Thermal post-buckling and large amplitude vibration analysis of Timoshenko beams are presented in paper [9]. Solution is shown in the form of simple closed-form solutions by making use of the Rayleigh–Ritz method. Proposed approach is compared to the available results in the literature. Thermal buckling and elastic vibrations of functionally graded beams are under consideration in the work [10]. The Ritz method is adopted to solve the eigenvalue problems that are associated with thermal buckling and vibration in various types of immovable boundary conditions.

This paper presents the formulation of the boundary problem of column vibration, which is additionally exposed to a local heat source. This issue is limited to a linear range. As a result of thermal load, the material properties (e.g. Young's modulus) of the system under consideration change. By solving the boundary - initial heat flow problem, distributions of average values of Young's modulus are obtained in the next heating times. Using these results, the characteristic curves of the system depending on its diameter are determined for different times of exposure to the heat source.

2. The Initial-boundary problem of the heat flow

Transient heat transport in the column (Fig. 1a) is described by the following partial differential equation

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)=c\rho\frac{\partial T}{\partial t}, \tag{1}$$

where: λ is the thermal conduction coefficient [W/mK], c represents the specific heat [J/kgK], ρ denotes the density [kg/m^3], T represents the temperature [K], x, y, z are the Cartesian coordinates [m] and t denotes time [s].

Equation (1) is supplemented by the appropriate boundary and initial conditions:

$$q_b = -\lambda \frac{\partial T}{\partial \mathbf{v_n}}, T\left(x, y, z, t = 0\right) = T_0, \tag{2-3}$$

where: q_b represents the value of boundary heat flux [W/m^2], $\mathbf{v_n}$ is the direction of the vector perpendicular to the surface and T_0 represents the initial temperature [K].

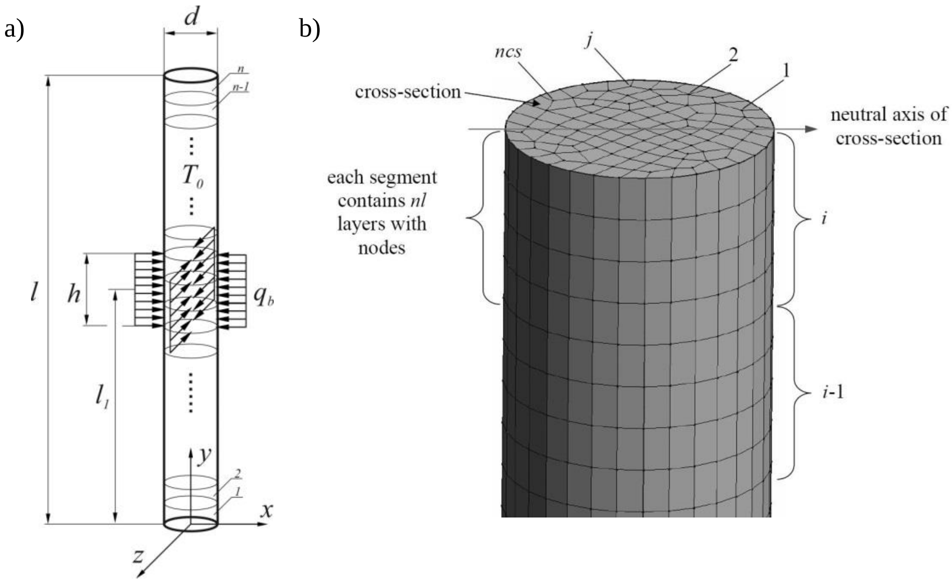

Figure 1. a) Scheme of the column with the thermal load b) fragment of the mesh

Equation (1) complemented by the conditions (2, 3) is solved by the means of FEM. Using the standard Galerkin formulation for the spatial discretization and then the procedure of aggregation of the finite elements and the Euler backward scheme of the integration with respect to time the final FEM equation is obtained:

$$\left(\mathbf{K} + \frac{1}{\Delta t}\mathbf{M}\right)\mathbf{T}^{f+1} = \mathbf{B} + \frac{1}{\Delta t}\mathbf{M}\mathbf{T}^{f}, \tag{4}$$

where: $\mathbf{K}$ is the matrix of thermal conductivity, $\mathbf{M}$ is the matrix of thermal capacity, $\mathbf{B}$ is the vector containing boundary conditions, Δt represents the time step [s], f is the time level.

Scheme (4) is used to compute the temperature distribution in the column in the each time step. The investigation of stability of the column is preceded by the division of the column into n segments (Fig. 1a). The Young's modulus is the function of temperature calculated according to [11-12]:

$$\frac{E(T_C)}{E(20)} = 1 + \frac{T_C}{2000\ln\left(\frac{T_C}{1100}\right)} \quad \text{for } 20^\circ C \le T_C \le 600^\circ C,$$
$$\frac{E(T_C)}{E(20)} = \frac{690 - 0.69T_C}{T_C - 53.5} \quad \text{for } 600^\circ C \le T_C \le 1000^\circ C, \tag{5}$$

where: T_C is the temperature expressed in degrees Celsius, $E(20)$ is the value of Young modulus for 20[°C] .

The mesh is composed of the horizontal layers with the same distribution of nodes (Fig. 1b). Each segment of the column contains n_l of nodal layers. Computation of the stiffness to expansion-compression $(EA)_{cs}$, bending stiffness $(EI)_{cs}$ and temperature T_{cs} in every cross section of the column is performed in the following way:

$$(EA)_{cs} = \sum_{j=1}^{n_{cs}} (EA)_j ,\ (EI)_{cs} = \sum_{j=1}^{n_{cs}} (EI)_j ,\ T_{cs} = \frac{\sum_{j=1}^{n_{cs}} A_j T_j}{A_{cs}}, \tag{6-8}$$

where: T_j is the temperature in the j-th node, E_j represents Young's modulus, A_j is the fragment of the cross-sectional area assigned to the j-th node, $I_j = A_j r_j^2$ is the moment of inertia, A_j, r_j – the distance between the neutral axis of the cross section and the j-th node, n_{cs} – the total number of nodes in the cross-sectional area, A_{cs} – the cross-sectional area of the column.

Averaged value of the parameters in the *i-th* segment are obtained as follows:

$$(EA)_i = \frac{\sum_{cs=1}^{n_l} (EA)_{cs}}{n_l},\ (EI)_i = \frac{\sum_{cs=1}^{n_l} (EI)_{cs}}{n_l},\ (T)_i = \frac{\sum_{cs=1}^{n_l} T_{cs}}{n_l}. \tag{9-11}$$

Above parameters are averaged for the segment which is composed of n_l horizontal layers of nodes. These nodes are vertices of quadrilateral finite elements which also form n_l cross sections in every segment of the column (Fig. 1b). *EA*, *EI* and *T* are initially calculated for every cross section (6-8) and then averaged for the segment (9-11). Such averaging is necessary for the boundary problem of free vibrations. Parameters (9-11) are finally saved in the text files for the next part of solution.

3. The boundary problem of free vibrations

The boundary problem of vibrations is formulated for the system presented in Figure 2. The column is considered to be articulated on both sides with a length of *1*, which has been divided into *n* equal segments. Such a division is made to obtain a distribution of the mean values of the Young's modulus along the length of the column. The system is loaded with force *P*, whose direction of action is constant and does not change with the column displacement. In addition, the system is subjected to thermal load q_b on a defined column surface with a height h and at a distance of l_q from the bottom mounting.

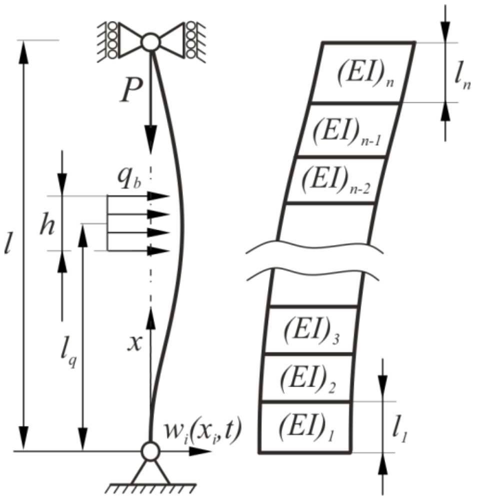

Figure 2. Scheme of the analysed system

The problem is formulated using the Hamilton principle (12). The potential energy V and kinetic energy T of the system can be written in the following form:

$$\delta\int_{t1}^{t2}\left(T-V\right)=0, \tag{12}$$

$$V=\frac{1}{2}\sum_{i=1}^{n}\left(EI\right)_i\int_0^{l_i}\left(\frac{d^2w\left(x\right)}{dx^2}\right)^2dx-\frac{1}{2}P\int_0^{l}\left(\frac{dw\left(x\right)}{dx}\right)^2dx, \tag{13}$$

$$T=\frac{1}{2}\rho A\int_0^{l}\left(\frac{\partial w\left(x,t\right)}{\partial t}\right)^2dx. \tag{14}$$

The geometric boundary conditions and geometric continuity conditions are as follows:

$$w_1\left(0,t\right)=0,\, w_n\left(l_n,t\right)=0,\, w_i\left(l_i,t\right)=w_{i+1}\left(0,t\right), \tag{15-17}$$

$$\left.\frac{\partial w_i\left(x_i,t\right)}{\partial x_i}\right|^{x_i=l_1}=\left.\frac{\partial w_{i+1}\left(x_i,t\right)}{\partial x_i}\right|_{x_1=0}. \tag{18}$$

After applying the Hamilton principle and after performing appropriate mathematical transformations, differential equations of motion (19) and natural boundary conditions for individual system segments (20-23) are obtained:

$$(EI)_i \frac{\partial^4 w_i(x_i,t)}{\partial x_i^{\,4}} + P\frac{\partial^2 w_i(x_i,t)}{\partial x_i^{\,2}} + \rho A \frac{\partial^2 w_i(x_i,t)}{\partial t^2} = 0, \tag{19}$$

$$\left.\frac{\partial^2 w_1(x_1,t)}{\partial x_1^{\,2}}\right|_{x_1=0} = 0, \left.\frac{\partial^2 w_n(x_n,t)}{\partial x_n^{\,2}}\right|^{x_n=l_n} = 0, \tag{20-21}$$

$$(EI)_i \left.\frac{\partial^2 w_i(x_i,t)}{\partial x_i^{\,2}}\right|^{x_i=l_1} = (EI)_{i+1} \left.\frac{\partial^2 w_{i+1}(x_i,t)}{\partial x_i^{\,2}}\right|_{x_1=0}, \tag{22}$$

$$(EI)_i \left.\frac{\partial^3 w_i(x_i,t)}{\partial x_i^{\,3}}\right|^{x_i=l_1} = (EI)_{i+1} \left.\frac{\partial^3 w_{i+1}(x_i,t)}{\partial x_i^{\,3}}\right|_{x_1=0}. \tag{23}$$

In equations (15-23), the operation of separating variables is carried out, assuming:

$$w_i(x_i,t) = Y_i(x_i)\cos\omega t, \tag{24}$$

The solution of the equation of motion is assumed in the form of:

$$Y_i(x_i) = A_i \cosh(\alpha_i x_i) + B_i \sinh(\alpha_i x_i) + C_i \cos(\beta_i x_i) + D_i \sin(\beta_i x_i), \tag{25}$$

where:

$$\alpha_i = \sqrt{-\frac{k_i^{\,2}}{2} + \sqrt{\frac{k_i^{\,4}}{4} + \Omega_i^{\,2}}}, \beta_i = \sqrt{\frac{k_i^{\,2}}{2} + \sqrt{\frac{k_i^{\,4}}{4} + \Omega_i^{\,2}}}, \tag{26-27}$$

$$k_i^{\,2} = \frac{P}{(EI)_i}, \Omega_i^{\,2} = \frac{\rho A \omega^2}{(EI)_i}. \tag{28-29}$$

Substituting the solution (25) to the boundary conditions (15-18 and 20-23), a system of equations is obtained whose determinant of the matrix of coefficients compared to zero is a transcendental equation, used to determine the natural frequency of the system.

4. Results of numerical simulations

The results of numerical calculations are presented in the form of characteristic curves. The following dimensionless parameters were used to relate the results to a heat-free system:

$$\lambda = \frac{Pl^2}{(EI)_0}, \Omega^* = \frac{\rho A \omega^2 l^4}{(EI)_0}, \tag{30-31}$$

where $(EI)_0$ is the product of moment of inertia and Young modulus in temperature equal to 20[°C].

Calculations are carried out for a column with a length l = 1 [m], loaded with a heat flux q_b = 200 000 [W/m^2]. The column is thermally loaded in half its height (l_q = 0.5 [m]). The height of the heat source h is = 0.2 [m].

Fig. 3 presents characteristic curves for four variants of column diameters (d = 0.01; 0.015; 0.02; 0.025 [m]). In each case, calculations are made for subsequent exposure times to the heat source (with a time interval of 15 [s]). The characteristics are linear. Due to the adopted dimensionless parameters, the curves in the charts marked with the number 1 have the same form - they refer to a system that is not thermally loaded. Based on the calculations, it is found that as the heat load time increases, the characteristic curves move towards lower values. This results in a decrease in both the critical load of the system (in the case of the divergential system, the critical load corresponds to the case of Ω^* = 0) and the natural frequency. The effect of heat load on column vibration is non-linear. Due to the assumed constant value of the heat load, along with the increase in the diameter of the system, the time after which the influence of the heat load has a significant effect on the system's vibration increases. For this reason, on the following charts for systems with higher diameters, the exposure time is greater.

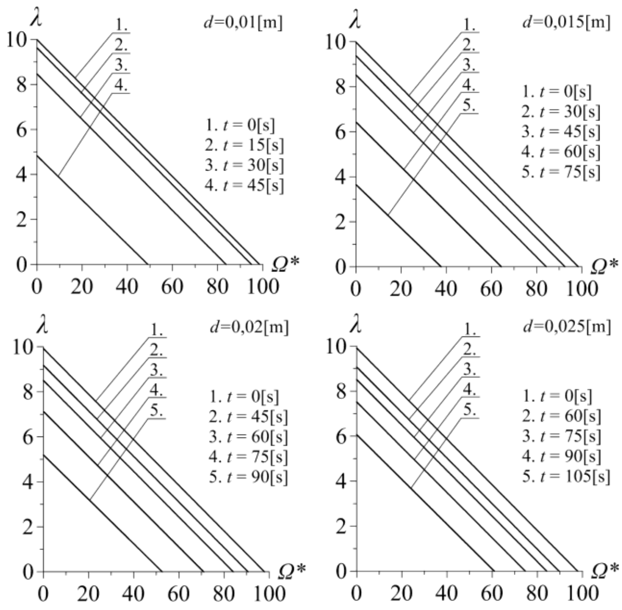

Figure 3. Impact of the heating time on the characteristic curves for different diameters of the analysed system

5. Conclusions

In this work, formulation of the boundary problem of column vibration exposed to Euler load and heat load was carried out. The issue of heat flow in the column was solved using FEM. Based on the Young's modulus distributions in the column for subsequent heating times, numerical simulations were performed regarding linear vibrations of the system. The effect of column heating time on characteristic curves was determined, which in the case of slender support systems subjected to compression are of great importance in the process of designing support structures. The results are presented for various system diameters and different heating times. This problem will be developed in the future, in particular to include the stiffness of the supported system (the column can be treated as a slender support system) and to extend the formulation to the issue of non-linear vibrations.

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