Vibrations of Microstructured Beams with Axial Force

Jarosław JĘDRYSIAK
Department of Structural Mechanics, Łódź University of Technology,
al. Politechniki 6, 90-924 Łódź, Poland,
jarek@p.lodz.pl

Abstract

In this contribution there are considered vibrations of microstructured periodic slender beams, with axial force. In order to analyse the effect of the microstructure size of the beams on their vibrations the tolerance modelling method is applied. Using this method there are derived governing equations of two tolerance models – general and standard, base on two various concepts – weakly-slowly-varying functions and slowly-varying functions. These models are applied to obtain formulas of lower order and higher order frequencies with influence of the axial force. To evaluate these results of the modelling the formula of lower order frequencies in the framework of the asymptotic model (neglecting the effect of the microstructure size) is also derived.

Keywords: microstructured periodic beams, slender beams, effect of microstructure, tolerance modelling.

1. Preliminaries

The subject of this note is an analysis of linear vibrations of slender periodic beams with axial force. The governing equation of these beams is determined by a partial differential equation with highly oscillating, periodic, non-continuous coefficients. This equations is not a good tool to investigate vibration problems. Hence, various averaged models are proposed to obtain substitute governing equations.

Different mechanical problems of beams or plates with microstructure were analysed in a series of papers, e.g.: a homogenization with microlocal parameters was applied to model microperiodic plates in [1]; an asymptotic homogenization method was used to analyse bending of periodic beams in [2]; a certain mathematical modelling for dynamic stability of sandwich beams with a microstructured was proposed in [3]; torsion of a composite bar made of an auxetic material with a microstructure was considered in [4, 5]. However, the effect of the microstructure size is neglected in equations of the models. But this effect can play a role on the general behaviour of microstructured media, cf. [6, 7], where higher order vibrations related to the microstructure were analysed. Various analytical and numerical model were proposed for similar problems of periodic beams, e.g.: a differential quadrature method was used in [8, 9]; a transfer matrix method was considered in [10, 11]; a multi-reflection method was applied in [12].

In this paper, in order to take into account the effect of the microstructure in governing equations of the model, the tolerance averaging technique, called also the tolerance modelling method, is applied. This approach is used to obtain a mathematical model, which describes vibrations of periodic beams by partial differential equations with constant coefficients and allows to investigate the aforementioned effect. Applications of this method to different mechanical problems of microstructured media can be found in monographs cf. [13-15] and in a series of works, e.g.: dynamic problems of micro-periodic beams were modelled in [16]; vibrations of periodic wavy-plates were considered in [17]; thin plates reinforced by a periodic system of stiffeners were analysed in [18]; a dynamics of medium-thickness periodic plates was investigated in [19]; an analysis of vibrations of thin periodic plates was shown in [20]; a problem of dynamic stability for micro-periodic cylindrical shells was presented in [21]; nonlinear vibrations of periodic beams were analysed in [22].

In this work averaged governing equations of the linear tolerance model for vibrations of slender periodic beams with axial force are presented and applied to obtain frequencies of lower and higher order vibrations for simply supported beams.

2. Foundations

In considered slender periodic beams cross-sections and material properties can change periodically along their longitudinal axes.

Let us introduce an orthogonal Cartesian coordinate system *Oxyz* and the axis of the beam *Ox*. Denote sizes of the cross section of the beam along *z* and *y*-axis as the height *h* and the width *b*, respectively. The undeformed beam occupies the region denoted by $\Omega \equiv \{(x,y,z): -b/2 \le y \le b/2, -h/2 \le z \le h/2, x \in \Lambda\}$, with the beam axis Λ, $\Lambda \equiv [0,L]$. Derivatives of x are denoted by ∂; the "basic cell" on *Ox* by $\Delta \equiv [-l/2, l/2]$, with l as the length of cell, being the period of inhomogeneity and called *the microstructure parameter*. This parameter satisfies the condition $h_{\max} << l << L$. It is assumed that geometrical properties of the beam: height $h(\cdot)$ and width $b(\cdot)$ can be periodic functions in x, but material properties of the beam: modulus of elasticity $E=E(\cdot,y,z)$, mass density $\rho=\rho(\cdot,y,z)$ can be periodic functions in x and even functions in y, z. The deflection of the beam is denoted by $w=w(x,t)$ and p be total loadings in the z-axis direction.

Using the modelling assumptions of the slender beams theory: the kinematic assumption of slender beams, the strain-displacement relation, the stress-strain relation, the virtual work equation can be formulated. After some manipulations and introducing denotations for periodic functions in x – bending stiffness $d(\cdot)$, mass density $\mu(\cdot)$, rotational mass inertia $j(\cdot)$:

$$\begin{aligned} &d(x) = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} E(x,y,z) z^2 dz dy, \\ &\mu(x) = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} \rho(x,y,z) dz dy, \qquad j(x) = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} \rho(x,y,z) z^2 dz dy, \end{aligned} \tag{1}$$

and also for the axis force:

$$n = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} s_0 dz dy, \tag{2}$$

with s_0 as the initial stress, the following known governing equation of slender periodic beams can be written:

$$\partial\partial(d\partial\partial w) - \partial(n\partial w) + \mu\ddot{w} - j\partial\partial\ddot{w} = p\ . \tag{3}$$

Equation (3) has highly oscillating, periodic, non-continuous coefficients.

3. Outline of tolerance modelling procedure

To obtain the averaged governing equations of slender periodic beams the introductory concepts and modelling assumptions of the tolerance modelling method are used, cf. [15]. Here, these concepts are mentioned: the tolerance system, a tolerance parameter, an averaging operator <·>, a tolerance-periodic function (*TP*), a slowly-varying function (*SV*), a weakly-slowly-varying function (*WSV*), a highly oscillating function (*HO*), a fluctuation shape function (*FS*). Different tolerance models can be derived applying concepts of: the weakly-slowly-varying function or the slowly-varying function.

Using the aforementioned concepts the modelling assumptions of the tolerance modelling method can be formulated. Below, these assumptions are reminded following the book [15] as:

- *the micro-macro decomposition* of the unknown deflection of the beam in the form:

$$w(x,t) = W(x,t) + g^A(x)Q^A(x,t), \quad A = 1,\ldots,M, \quad x \in \Lambda, \tag{4}$$

with: new kinematic unknowns named *the macrodeflection* $W(\cdot,t)$ and *the fluctuation amplitudes* $Q^A(\cdot,t)$, being $W(\cdot,t), Q^A(\cdot,t) \in WSV$ or $W(\cdot,t), Q^A(\cdot,t) \in SV$, i.e. weakly-slowly- or slowly-varying functions; the known fluctuation shape functions $g^A(\cdot)$, being usually postulated *a priori* in the case under consideration and describing the unknown deflection oscillations caused by the beam periodicity;

- *the tolerance averaging approximation*, where it is assumed that terms $O(\delta)$ are negligibly small, e.g. for $f \in TP$, $F \in SV$ or $F \in WSV$, $g^A \in FS$, in:

$$\begin{aligned} &< f > (x) =< \bar{f} > (x) + O(\delta), \\ &< fF > (x) =< f > (x)F(x) + O(\delta), \\ &< f\partial(g^A F) > (x) =< f\partial g^A > (x)F(x) + O(\delta); \end{aligned} \tag{5}$$

with δ being the tolerance parameter;

- *the axis force restriction*, where terms involving fluctuating parts of the axis force are assumed that can be neglected in comparing to terms with averaged parts, i.e.:

$$\begin{aligned} &n(x) = N(x) + \tilde{n}(x), \\ &N =< n >, \qquad\qquad < \tilde{n} >= 0, \end{aligned} \tag{6}$$

with $N(\cdot) \in SV$ and $\tilde{n}(\cdot) \in TP$ being *averaged* and *fluctuating part of axis force*, respectively.

After the book [15] the tolerance modelling procedure can be outlined as follows.

In the first step the micro-macro decomposition (4) is substituted into equation (3). Since after this dynamic equation (3) does not hold, therefore, there should be a residual field $r(\cdot)$ within macrodynamics as follows:

$$\begin{aligned} r = \partial\partial(d\partial\partial(W + g^A Q^A)) - \partial(n\partial(W + g^A Q^A)) + \\ +\mu(\ddot{W} + g^A \ddot{Q}^A) - j\partial\partial(\ddot{W} + g^A \ddot{Q}^A) - p. \end{aligned} \tag{7}$$

In the second step the assumption called as *the residual orthogonality condition*, in which the residual field $r(\cdot)$ satisfies the conditions:

$$< r > (x,t) = 0, \qquad\qquad < rg^B > (x,t) = 0; \tag{8}$$

is used to the formula (7), together with assumptions (5), (6).

After some manipulations in the third step a system of equations for the macrodeflection $W(\cdot,t)$ and the fluctuation amplitudes $Q^A(\cdot,t)$, A=1,...,M, is obtained, the form of which is related to the specification of the class of slowly-varying functions $W(\cdot,t)$, $Q^A(\cdot,t)$ (weakly-slowly-varying or slowly-varying functions).

4. Governing equations of models

After applying the above modelling procedure with the concept of weakly-slowly-varying function and introducing the following denotations of averaged coefficients:

$$
\begin{array}{lll}
D \equiv < d >, & D^A \equiv < d\partial\partial g^A >, & D^{AB} \equiv < d\partial\partial g^A \partial\partial g^B >, \\
\tilde{m} \equiv < \mu >, & \tilde{m}^{AB} \equiv l^{-4} < \mu g^A g^B >, & \\
\vartheta \equiv < j >, & \vartheta^{AB} \equiv l^{-2} < j\partial g^A \partial g^B >, & \\
N \equiv < n >, & N^{AB} \equiv l^{-2} < n\partial\partial g^A g^B >, & \breve{N}^{AB} \equiv -l^{-2} < n\partial g^A \partial g^B > = -N^{AB}, \\
P \equiv < p >, & P^A \equiv l^{-2} < p g^A >; & \\
\overline{N}^A \equiv l^{-2} < n g^A >, & \overline{N}^{AB} \equiv l^{-4} < n g^A g^B >, & \overline{D}^A \equiv l^{-2} < d g^A >, \\
\overline{D}^{AB} \equiv l^{-4} < d g^A g^B >, & \hat{D}^{AB} \equiv l^{-2} < d\partial g^A \partial g^B >, & \breve{D}^{AB} \equiv l^{-2} < d\partial\partial g^A g^B >, \\
\overline{\vartheta}^A \equiv l^{-2} < j g^A >, & \overline{\vartheta}^{AB} \equiv l^{-4} < j g^A g^B >; &
\end{array}
\tag{9}
$$

the system of equations in the form is derived:

$$
\begin{array}{l}
\partial\partial(D\partial\partial W + D^A Q^A + \underline{l^2 \overline{D}^A \partial\partial Q^A}) - \partial(N\partial W) + \tilde{m}\ddot{W} - \vartheta\partial\partial\ddot{W} - \\
-\underline{l^2\partial(\overline{N}^A \partial Q^A + \overline{\vartheta}^A \partial \ddot{Q}^A)} = P, \\
D^A\partial\partial W + D^{AB}Q^B + l^2(l^2 m^{AB} + \vartheta^{AB})\ddot{Q}^B - l^2 N^{AB} Q^B - \underline{l^2 \overline{N}^A \partial\partial W} + \\
+\underline{l^2\partial\partial(\overline{D}^A \partial\partial W + l^2 \overline{D}^{AB}\partial\partial Q^B)} + \underline{l^2[2(\breve{D}^{AB} - 2\hat{D}^{AB}) - l^2 \overline{N}^{AB}]\partial\partial Q^B} - \\
-\underline{l^2\partial(\overline{\vartheta}^A \partial\ddot{W} + l^2 \overline{\vartheta}^{AB}\partial\ddot{Q}^B)} = l^2 P^A,
\end{array}
\tag{10}
$$

which stands the governing equations of *the general tolerance model of slender periodic beams*. The basic unknowns are: *the macrodeflection W* and *the fluctuation amplitudes* Q^A, A=1,…,M, which are weakly-slowly-varying functions in x. Equations (10) have constant coefficients, some of which (underlined) depend on the microstructure parameter l. Thus, they describe the effect of the microstructure size on the overall behaviour of the considered beams.

Using the above modelling procedure presented in Section 3 with the concept of slowly-varying function and the denotations of averaged coefficients (9), the following equations are obtained:

$$
\begin{array}{l}
\partial\partial(D\partial\partial W + D^A Q^A) - \partial(N\partial W) + \tilde{m}\ddot{W} - \vartheta\partial\partial\ddot{W} = P, \\
D^A\partial\partial W + D^{AB}Q^B + l^2(l^2 m^{AB} + \vartheta^{AB})\ddot{Q}^B - l^2 N^{AB} Q^B = l^2 P^A.
\end{array}
\tag{11}
$$

They are the governing equations of *the (standard) tolerance model of slender periodic beams*. The basic unknowns of them: *the macrodeflection W* and *the fluctuation amplitudes* Q^A, A=1,…,M, are slowly-varying functions in x. Similarly to Equations (10) the above equations (11) have also constant coefficients, some of which depend on the microstructure parameter l. They allow also to analyse the effect of the microstructure size on the overall behaviour of the considered beams. It can be observed that Equations (11) can be also obtained from Equations (10) after neglecting underlined terms.

In order to evaluate both the tolerance models the averaged model without the effect of the microstructure size is formulated, i.e. its governing equations have not coefficients dependent on the microstructure parameter l. Equations of this model can be obtained using an asymptotic modelling procedure, cf. [15], or from Equations (11) after neglecting terms with the microstructure parameter l. These equations take the form:

$$\begin{aligned}&\partial\partial(D\partial\partial W + D^A Q^A) - \partial(N\partial W) + \tilde{m}\ddot{W} - \vartheta\partial\partial\ddot{W} = P,\\ &D^A\partial\partial W + D^{AB}Q^B = 0.\end{aligned} \tag{12}$$

They represent *the asymptotic model of slender periodic beams*. The above equations have constant coefficients, similarly to Equations (10) or (11) for the tolerance models, in the contrast to Equation (3) with non-continuous, highly oscillating, periodic, functional coefficients. Moreover, they vanish the effect of the microstructure size on the overall behaviour of the beams.

5. Example – free vibrations of periodic beams with axial force

As an example a simply supported slender periodic beam is considered. It is assumed that: load p=0; axial force n=const; the geometrical properties of the beam – height h and width b are constant; the material properties – Young's modulus E and Poisson's ratio ν are also constant. The periodicity of the beam is caused by the periodic distribution of its mass density defined as:

$$\rho(\cdot,\xi) = \begin{cases}\rho', & \text{for} \quad \xi \in ((1-\gamma)l/2, (1+\gamma)l/2),\\ \rho'', & \text{for} \quad \xi \in [0,(1-\gamma)l/2] \cup [(1+\gamma)l/2, l],\end{cases} \tag{13}$$

with a distribution parameter γ of material properties.

The introductory concept as the fluctuation shape functions g^A, A=1,…,M, play important role in the modelling analysis, because they describe oscillations of the beam deflection in the periodicity cell. Here, it is assumed only one fluctuation shape function, i.e. A=M=1, g=g^1, in the following form related to the symmetric periodicity cell:

$$g(x) = l^2[\sin(2\pi x/l) + c], \tag{14}$$

with the constant c calculated from the condition $<\mu g>=0$.

Hence, introducing denotations:

$$\begin{aligned}&d = \tfrac{1}{12}Ebh^3; && H^{11} = <\partial\partial g\partial\partial g>,\\ &G^{11} = l^{-2} <\partial\partial gg>, && \breve{G}^{11} = l^{-2} <\partial g\partial g> = -G^{11};\\ &\overline{G}^1 = l^{-2} <g>, && \overline{G}^{11} = l^{-4} <gg>;\end{aligned} \tag{15}$$

the averaged coefficients defined by (9) and different from zero for the above assumed properties can be written as:

$$\begin{aligned}&D = d, && D^{11} = dH^{11}, && \tilde{m} = <\mu>, && \tilde{m}^{11} = l^{-4} <\mu gg>,\\ &\vartheta = <j>, && \vartheta^{11} = l^{-2} <j\partial g\partial g>; && \overline{\vartheta}^1 = l^{-2} <jg>, && \overline{\vartheta}^{11} = l^{-4} <jgg>;\\ &N = n, \quad N^{11} = nG^{11}, && \breve{N}^{11} = -n\breve{G}^{11} = -nG^{11}; && \overline{N}^1 = n\overline{G}^1, && \overline{N}^{11} = n\overline{G}^{11},\\ &\overline{D}^1 = d\overline{G}^1, && \overline{D}^{11} = d\overline{G}^{11}, && \hat{D}^{11} = -d\breve{G}^{11} = dG^{11}, && \breve{D}^{11} = dG^{11}.\end{aligned} \tag{16}$$

The governing equations of the presented averaged models take the following form for free vibrations with an axial force:

- *the general tolerance model*, (10):

$$\begin{aligned}
&d\partial\partial\partial\partial W+\underline{l^2 d\bar{G}^1\partial\partial\partial\partial Q}-n\partial\partial W+\tilde{m}\ddot{W}-\vartheta\partial\partial\ddot{W}-\underline{l^2(n\bar{G}^1\partial\partial Q+\bar{\vartheta}^1\partial\partial\ddot{Q})}=0,\\
&dH^{11}Q+l^2(l^2m^{11}+\vartheta^{11})\ddot{Q}-l^2nG^{11}Q-\underline{l^2n\bar{G}^1\partial\partial W}+\\
&+\underline{l^2d\partial\partial(\bar{G}^1\partial\partial W+l^2\bar{G}^{11}\partial\partial Q)}+\underline{l^2[2d(G^{11}-2\breve{G}^{11})-l^2n\bar{G}^{11}]\partial\partial Q}-\\
&-\underline{l^2\bar{\vartheta}^1\partial\partial\ddot{W}-l^4\bar{\vartheta}^{11}\partial\partial\ddot{Q}}=0;
\end{aligned} \tag{17}$$

- *the standard tolerance model*, (11):

$$\begin{aligned}
&d\partial\partial\partial\partial W-n\partial\partial W+\tilde{m}\ddot{W}-\vartheta\partial\partial\ddot{W}=0,\\
&dH^{11}Q+l^2(l^2m^{11}+\vartheta^{11})\ddot{Q}-l^2nG^{11}Q=0;
\end{aligned} \tag{18}$$

- *the asymptotic model*, (12):

$$\begin{aligned}
&d\partial\partial\partial\partial W-n\partial\partial W+\tilde{m}\ddot{W}-\vartheta\partial\partial\ddot{W}=0,\\
&dH^{11}Q=0.
\end{aligned} \tag{19}$$

It can be observed, that the considered problem of free vibrations is described in the framework of the general tolerance model by the system of coupled equations (17), in the standard tolerance model – by uncoupled two equations (18), and in the asymptotic model – by only one equation $(19)_1$.

After introducing the wave number k (e.g. k=2π/L), solutions to the above equations (17), (18), (19) have to satisfy the boundary conditions of a simply supported beam and can be assumed as:

$$W(x,t)=A_W\sin(kx)\cos(\omega t),\qquad Q(x,t)=A_Q\sin(kx)\cos(\omega t), \tag{20}$$

where: ω is a frequency; A_W and A_Q are amplitudes on the unknowns.

Substituting the solutions (20) into equations (17), (18), (19) the characteristic equations of free vibrations with the effect of the axial force can be derived for every presented above averaged model. From these characteristic equations the formulas of free vibration frequencies can be obtained.

Let us introduce the following denotations:

$$\begin{aligned}
&\alpha_1=k^2(k^2d+n),\qquad \alpha_2=\bar{G}^1l^2k^2(k^2d+n),\\
&\alpha_3=H^{11}d-l^2G^{11}n+l^2k^2\bar{G}^{11}(l^2k^2d+n)-2l^2k^2d(G^{11}-2\breve{G}^{11}),\\
&\beta_1=\tilde{m}+k^2\vartheta,\qquad \beta_2=l^2k^2\bar{\vartheta}^1,\qquad \beta_3=l^2(l^2m^{11}+\vartheta^{11}+l^2k^2\bar{\vartheta}^{11});\\
&\tilde{\alpha}_3=H^{11}d-l^2G^{11}n,\qquad \tilde{\beta}_3=l^2(l^2m^{11}+\vartheta^{11});
\end{aligned} \tag{21}$$

and:

$$a_2=\beta_1\beta_3-\beta_2^{\,2},\quad a_1=\alpha_1\beta_3+\alpha_3\beta_1-2\alpha_2\beta_2,\ a_0=\alpha_1\alpha_3-\alpha_2^{\,2}. \tag{22}$$

Using denotations (21) and (22) the formulas of free vibration frequencies for the considered beams can be written in the form:

- for *the general tolerance model*:

$$\omega_-=\sqrt{\frac{a_1-\sqrt{a_1^{\,2}-4a_0a_2}}{2a_2}},\qquad \omega_+=\sqrt{\frac{a_1+\sqrt{a_1^{\,2}-4a_0a_2}}{2a_2}}, \tag{23}$$

where: ω_- is *the fundamental lower order free vibration frequency* related to the beam macrostructure, ω_+ is the additional *higher order free vibration frequency* related to the beam microstructure;

- for *the standard tolerance model*:

$$\omega_0 = \sqrt{\frac{\alpha_1}{\beta_1}}, \qquad \omega_1 = \sqrt{\frac{\tilde{\alpha}_3}{\tilde{\beta}_3}}; \tag{24}$$

where: ω_0 is *the fundamental lower order free vibration frequency* related to the beam macrostructure, ω_1 is the additional *higher order free vibration frequency* related to the beam microstructure;

- for *the asymptotic model*:

$$\omega_0 = \sqrt{\frac{\alpha_1}{\beta_1}}. \tag{24}$$

where: ω_0 is *the fundamental lower order free vibration frequency* related to the beam macrostructure.

6. Conlusions and remarks

Summarizing the above analytical considerations following remarks can be formulated.

1. Using the tolerance modelling method the classic equation of slender periodic beams having periodic, non-continuous coefficients is replaced by the system of averaged equations of tolerance models with constant coefficients.
2. A choose of the class of basic unknown functions determines the form of the model equations. For the weakly-slowly-varying functions the equations of *the general tolerance model* are derived, which have additional terms with the microstructure parameter; but for the slowly-varying functions the equations of *the standard tolerance model* are obtained.
3. Both the tolerance models allow to investigate dynamic problems of the considered beams at the macro- and the micro-level, since their governing equations have terms describing the effect of the microstructure size.
4. In contrast, in the framework of *the asymptotic model* dynamic problems of considered beams can be analysed only at the macro-level, without the effect of the microstructure size.
5. The example allows to observe that the influence of the macrostructure appears as the macrovibrations (the lower order vibrations), which can be investigated within both the tolerance models and the asymptotic model. However, the effect of the microstructure is manifested as the microvibrations (the higher order vibrations) only in the framework of the tolerance models.

In forthcoming papers some other applications of the equations of the general and standard tolerance models for the slender periodic beams will be shown.

References

1. S.J. Matysiak, W. Nagórko, *Microlocal parameters in the modelling of microperiodic plates*, Ing. Arch., 59 (1989) 434-444.
2. A. Kolpakov, *Application of homogenization method to justification of 1-D model for beam of periodic structure having initial stresses*, Int. J. Solids Struct., 35 (1998) 2847-2859.

3. M. Grygorowicz, E. Magnucka-Blandzi, *Mathematical modeling for dynamic stability of sandwich beam with variable mechanical properties of core*, Appl. Math. Mech., 37 (2016) 361-1374.
4. T. Strek, H. Jopek, A. Fraska, *Torsion of elliptical composite beams with auxetic phase*, Phys. Status Solidi Basic Res., 253 (2016) 1359–1368.
5. H. Jopek, T. Stręk, *Torsion of a two-phased composite bar with helical distribution of constituents*, Phys. Status Solidi, 254 (2017) 1700050.
6. L. Brillouin, *Wave propagation in periodic structures*, Dover Pub. Inc., Dover, UK, 1953.
7. C.W. Robinson, G.W. Leppelmeier, *Experimental verification of dispersion relations for layered composites*, J. Appl. Mech., 41 *(*1974) 89-91.
8. H.-J. Xiang, Z.-F. Shi, *Analysis of flexural vibration band gaps in periodic beams using differential quadrature method*, Comp. Struct., 87 (2009) 1559-1566.
9. Z.B. Cheng, Y.G. Xu, L.L. Zhang, *Analysis of flexural wave bandgaps in periodic plate structures using differential quadrature element method*, Int. J. Mech. Sci., 100 (2015) 112–125.
10. D. Yu, J. Wen, H. Shen, Y. Xiao, X. Wen, *Propagation of flexural wave in periodic beam on elastic foundations*, Physics Letters A, 376 (2012) 626-630.
11. Y. Xu, X. Zhou, W. Wang, L. Wang, F. Peng, B. Li, *On natural frequencies of non-uniform beams modulated by finite periodic cells*, Physics Letters A, 380 (2016) 3278-3283.
12. T. Chen, *Investigations on flexural wave propagation of a periodic beam using multi-reflection method*, Arch. Appl. Mech., 83 (2013) 315-329.
13. C. Woźniak, E. Wierzbicki, *Averaging techniques in thermomechanics of composite solids. Tolerance averaging versus homogenization,* Techn. Univ. of Czestochowa Press, Częstochowa, Poland, 2000.
14. C. Woźniak, J. Jędrysiak, B. Michalak, *Thermomechanics of heterogeneous solids and structures, tolerance averaging approach*, Lodz Univ. of Techn., Łódź, Poland, 2008.
15. C. Woźniak (eds.), *Mathematical Modelling and Analysis in Continuum Mechanics of Microstructured Media*, Silesian Univ. Press, Gliwice, Poland 2010.
16. K. Mazur-Śniady, *Macro-dynamic of micro-periodic elastic beams*, J. Theor. Appl. Mech., 31 (1993) 781–793.
17. B. Michalak, *The meso-shape functions for the meso-structural models of wavy-plates*, ZAMM, 81 (2001) 639-641.
18. W. Nagórko, C. Woźniak, *Nonasymptotic modelling of thin plates reinforced by a system of stiffeners*, Electr. J. Polish Agric. Univ. – Civil Engineering, 5 (2002).
19. E. Baron, *On dynamic behaviour of medium-thickness plates with uniperiodic structure*, Arch. Appl. Mech., 73 (2003) 505-516.
20. J. Jędrysiak, *Free vibrations of thin periodic plates interacting with an elastic periodic foundation*, Int. J. Mech. Sci., 45 (2003) 1411-1428.
21. B. Tomczyk, *Dynamic stability of micro-periodic cylindrical shells*, Mech. Mech. Eng., 14 (2010) 137-150.
22. Ł. Domagalski, J. Jędrysiak, *Nonlinear vibrations of periodic beams*, J. Theor. Appl. Mech., 54 (2016) 1095–108.