# The Vibrations of Non-Symmetric Periodic Sandwich Beams

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**Abstract**

Sandwich structures are certain specific type of composites, which are widely used in modern engineering. In this paper the vibration analysis of a specific type of sandwich beam is performed. The considered structure is not only non-symmetric towards its midplane, but also made of periodically varying isotropic materials. As a result, the governing equations of such complicated structure is characterised by periodic, non-continuous and highly oscillating coefficients. With the use of the tolerance averaging technique those equations are transformed into the form with constant coefficients. Eventually, a comparative simulations of free vibration analysis of several sandwich beams were conducted to verify the effectiveness and superiority of proposed calculation method over the FEM.

***Keywords:*** *sandwich beams, vibrations, tolerance averaging technique, periodic microstructure*

## 1. Introduction

One of the most crucial type of composites used nowadays is a sandwich structure. It is a large and diverse group of multi-layered composites, which outer layers are usually made of materials characterised by high mechanical properties, while the inner layers are usually a lightweight and/or porous materials. In literature, one can find many different modelling approaches towards such structures, such as: a broken line hypothesis, first and higher order deformation hypothesis, equivalent single layer theory or Zig-Zag hypothesis. Most of the aforementioned approaches are dedicated to sandwich structures, which are symmetric towards its midplane. In this paper, the modelling of a composite, which is not symmetric towards its midplane, which may be required by a specific engineering issue, is presented.

The modelling approach presented in this paper is based on the broken line hypothesis. It is shown, that by adjusting several assumptions, with the use of the mentioned hypothesis it is possible to describe the dynamic behaviour of three-layered not-symmetric sandwich beam. The analysis of similar structures was developed by Frostig and Shenhar [1], Magnucka-Blandzi et al. [2] or Chakrabarti and Bera [3], among others. However, the proposed models are not able of describing the behaviour of structures with a specific periodic microstructure, such as: periodically varying thickness and/or material properties of layers. In this paper, the modelling of such sandwich structures is presented.

In order to solve such issue, the tolerance averaging technique, developed by Woźniak [4,5], is used. The mentioned technique has a lot of applications in the structural dynamics, such as: vibrations of Timoshenko beams [6], medium thickness plates [7] or shell structures [8], as well as in other issues of composite engineering, such as a thermoelasticity problems [9].

In the following paper, the issue of vibration analysis of a specific sandwich beam is investigated and the obtained results will be compared with the FEM model. As a result of this analysis, the comparison of free vibration frequencies and a discussion of the correctness of the obtained results is performed.

## 2. Modelling foundations

Let us denote $0x_1x_2x_3$ as an orthogonal Cartesian coordinate system, where $x \equiv x_1$, $z \equiv x_3$ and $t$ as a time coordinate. The considered three-layered beam is assumed to have span $L_1$ along $x_1$-axis direction and a constant width $b$ along $x_2$-axis direction. The midplane of the core of the structure can be denoted as $\Pi \equiv [0, L_1]$. Moreover, let us assume, that the considered structure is not symmetric towards the core midplane, hence, both thickness and material properties of both outer layers can be different for a specific $x$ coordinate. By introducing $h_c(x)$ as the thickness of the core, $h_f^{\,1}(x)$ as a thickness of the upper outer layer and $h_f^{\,2}(x)$ as a thickness of the lower outer layer it is possible to denote the whole region occupied by undeformed beam as: $\Omega \equiv \{(x,z) : -h_c(x)/2 - h_f^{\,1}(x) \le z \le h_c(x)/2 + h_f^{\,2}(x), x \in \Pi\}$, cf. Fig. 1.

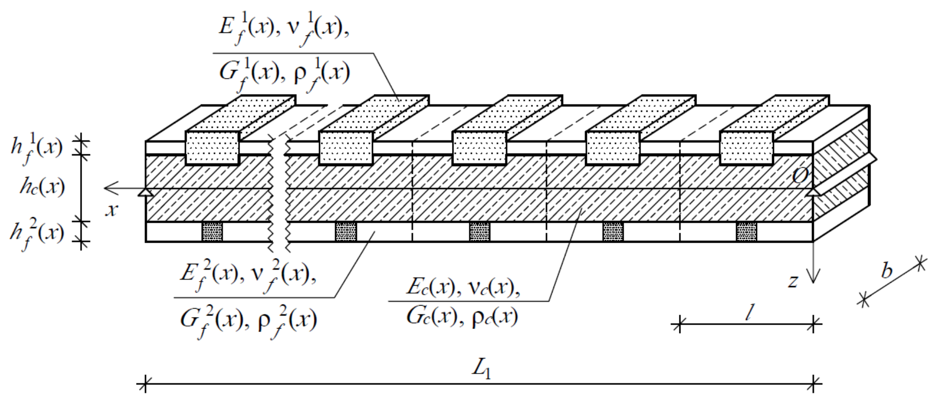


Figure 1. Sandwich three-layered beam with a certain periodic microstructure.

In all our considerations it is assumed that the whole structure is made of isotropic materials, hence let us introduce $E_c(x), \nu_c(x), G_c(x), \rho_c(x)$ as Young's modulus, Poisson's ratio, shear modulus and mass density of the core and $E_f^{\,i}(x), \nu_f^{\,i}(x), G_f^{\,i}(x), \rho_f^{\,i}(x), i = 1,2,$ as Young's modulus, Poisson's ratio, shear modulus and mass density of the outer layers, respectively, cf. Figure 1.

Every each layer of the considered three-layered beam can be characterised by a certain periodic microstructure, such as a periodically varying thickness and/or periodically varying material properties. By analysing the mentioned microstructure, it is possible to distinguish a small, repeatable element, called the periodicity cell Δ. In our case, let us assume, that the periodicity cell has dimension $l$ along $x$-axis direction, which will be referred to as *microstructure parameter*. In all subsequent sections it is assumed, that a spatial derivative is denoted as $\partial \equiv \frac{\partial}{\partial x}$ and a time derivative is denoted with an overdot.

### 3. The broken line hypothesis for non-symmetric sandwich beam

The initial point of all considerations is the formulation of the broken line hypothesis. Within this hypothesis the vertical displacements are assumed to be independent of the $z$ coordinate, while in-plane displacements of the considered beam is expressed with a specific, linear, piecewise function:

$$
\begin{aligned}
&u(x,z,t) = \begin{cases} F^1(x,z,t) & \text{for} \quad z \in < -h_f{}^1(x) - \tfrac{1}{2}h_c(x), -\tfrac{1}{2}h_c(x)) \\ F^c(x,z,t) & \text{for} \quad z \in < -\tfrac{1}{2}h_c(x), \tfrac{1}{2}h_c(x) > \\ F^2(x,z,t) & \text{for} \quad z \in (\tfrac{1}{2}h_c(x), \tfrac{1}{2}h_c(x) + h_f{}^2(x) > \end{cases}, \\
&w(x,z,t) \equiv w(x,t), \\
&F^1(x,z,t) \equiv u^1(x,t) - [z + \tfrac{1}{2}h_c(x) + \tfrac{1}{2}h_f{}^1(x)]\partial w(x,t), \\
&F^c(x,z,t) \equiv [z \cdot \frac{h_f{}^1(x) + h_f{}^2(x)}{2h_c(x)} - \tfrac{1}{4}h_f{}^1(x) + \tfrac{1}{4}h_f{}^2(x)]\partial w(x,t) + \\
&\qquad + z \cdot \frac{u^2(x,t) - u^1(x,t)}{h_c(x)} + \frac{u^2(x,t) + u^1(x,t)}{2}, \\
&F^2(x,z,t) \equiv u^2(x,t) - [z - \tfrac{1}{2}h_c(x) - \tfrac{1}{2}h_f{}^2(x)]\partial w(x,t),
\end{aligned} \tag{1}
$$

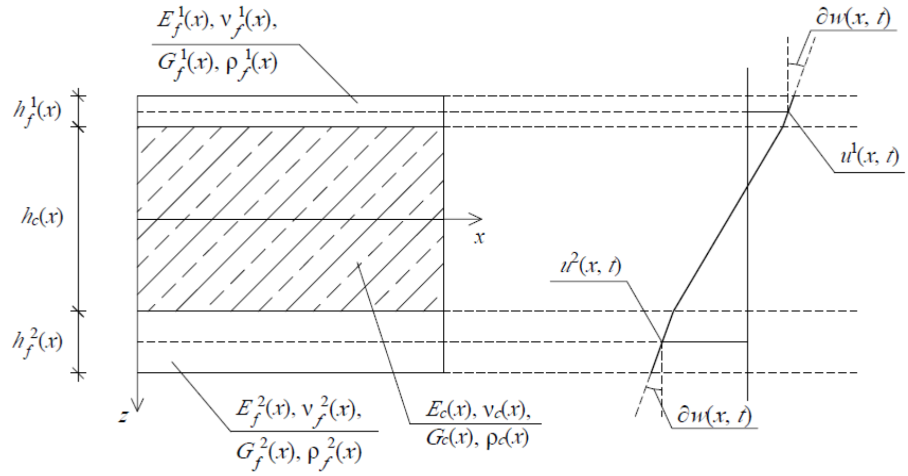


Figure 2. Broken line hypothesis for non-symmetric sandwich beam.

where $w(x,t)$ is a vertical displacement of the midplane of the core of the structure while functions $u^1(x,t), u^2(x,t)$ are in-plane displacements of the midplanes of the outer layers, respectively, cf. Figure 2.

Basing on the proposed functions, the equations of equilibrium on a basic, small part of the structure are investigated. By assuming, that the displacements of the structure are small, the stress-strain relation takes the form of classic Hooke's law for isotropic materials, and that the forces of inertia can act along both *x*- and *z*-axis directions, the initial system of governing equations can be formulated as:

$$
\begin{aligned}
&A_E\partial^3 w(x,t)+B^i\partial^2 u^i(x,t)-M\partial\ddot{w}(x,t)-M^i\ddot{u}^i(x,t)=0,\\
&C_E\partial^3 w(x,t)+\tilde{C}\partial w(x,t)+D^i\partial^2 u^i(x,t)+G_c[u^1(x,t)-u^2(x,t)]+\\
&\qquad +\tilde{M}\partial\ddot{w}(x,t)-\tilde{M}^i\ddot{u}^i(x,t)=0,\\
&C_E\partial^4 w(x,t)+D^i\partial^3 u^i(x,t)+\tilde{M}\partial^2\ddot{w}(x,t)-\tilde{M}^i\partial\ddot{u}^i(x,t)-\mu\ddot{w}(x,t)=-p_3(x,t),\\
&i=1,2,
\end{aligned}
\tag{2}
$$

where $p_3(x,t)$ is a vertical external loading, while the definitions of introduced coefficients are presented below:

$$
\begin{aligned}
&A_E=\tfrac{1}{4}E_c h_c({h_f}^2-{h_f}^1), \qquad B^i={E_f}^i{h_f}^i+\tfrac{1}{2}E_c h_c,\\
&C_E=-\tfrac{1}{12}[{E_f}^1({h_f}^1)^3-\tfrac{1}{2}E_c(h_c)^2({h_f}^1+{h_f}^2)+{E_f}^2({h_f}^2)^3],\\
&\tilde{C}=-G_c(h_c+\tfrac{1}{2}{h_f}^1+\tfrac{1}{2}{h_f}^2), \qquad D^i=(-1)^i\tfrac{1}{2}[{E_f}^i{h_f}^i(h_c+{h_f}^i)+\tfrac{1}{6}E_c(h_c)^2],\\
&M=\tfrac{1}{4}\rho_c h_c({h_f}^2-{h_f}^1), \qquad M^i=\tfrac{1}{2}\rho_c h_c+{\rho_f}^i{h_f}^i,\\
&\tilde{M}=\tfrac{1}{12}[{\rho_f}^1({h_f}^1)^3-\tfrac{1}{2}\rho_c(h_c)^2({h_f}^1+{h_f}^2)+{\rho_f}^2({h_f}^2)^3],\\
&\tilde{M}^i=(-1)^i[\tfrac{1}{12}\rho_c(h_c)^2+\tfrac{1}{2}{\rho_f}^i{h_f}^i(h_c+{h_f}^i)],\\
&\mu={\rho_f}^1{h_f}^1+\rho_c h_c+{\rho_f}^2{h_f}^2.
\end{aligned}
\tag{3}
$$

One should notice, that material properties and thickness of every each layer in denotations (3) can be functions of coordinate *x*. Hence, the presented system of initial governing equations of motion are equations with periodic, highly-oscillating and non-continuous coefficients, which are very difficult to solve using simple mathematical methods. In the next step of modelling all those equations will be transformed into the form with constant coefficients with the use of the tolerance averaging technique.

## 4. Averaged model of non-symmetric sandwich beam

The modelling procedure, which leads to the derivation of the averaged model, is based on the tolerance averaging technique. The precise description of all concepts of this technique can be found in literature, cf. Woźniak et al. [4,5].

There are two main assumptions of the tolerance averaging technique. The first of them is *the micro-macro decomposition*, according to which a specific physical field $u(\cdot,t)$ can be expressed as a sum of averaged macrofield of a certain physical property $U(\cdot,t)$ and a product of fluctuation-shape functions $h^A(\cdot)$ and their amplitudes $v^A(\cdot,t)$ :

$$u(\cdot,t) = U(\cdot,t) + h^A(\cdot)v^A(\cdot,t), \qquad A = 1,...,N. \tag{4}$$

Both macrofield $U(\cdot,t)$ and fluctuation amplitudes $v^A(\cdot,t)$ are assumed to be slowly varying functions for every *t*, which means, that they are almost constant on the basic periodicity cell $\Delta$.

The second assumption is a set of tolerance averaging approximations, using which the averaged terms can be simplified into the most convenient form. By introducing certain given a priori tolerance parameter $\delta$, it is possible to prove the relations:

$$\begin{aligned} &< \Phi > (x) =< \bar{\Phi} > (x) + O(\delta), \qquad < \Phi F > (x) =< \Phi > (x)F(x) + O(\delta), \\ &< \Phi \partial_\alpha (gF) > (x) =< \Phi \partial_\alpha g > (x)F(x) + O(\delta), \end{aligned} \tag{5}$$

where $< \cdot >$ is an averaging operator, $\Phi$ is tolerance-periodic function, $\bar{\Phi}$ is periodic approximation of $\Phi$, *F* is slowly varying function, *g* is highly oscillating function and $O(\delta)$ is negligibly small term, $0 < \delta << 1$, cf. Woźniak [4,5].

The whole tolerance modelling procedure consists of several steps, which includes the application of the averaging operator to the system of initial equations (2) and transformations with the use of both the micro-macro decomposition and the tolerance averaging approximations. In our case, the micro-macro decomposition of displacements fields can be introduced in the form:

$$w(x,t) = W(x,t) + h^A(x)Q^A(x,t), \quad u^i(x,t) = U^i(x,t) + g^{i,B}(x)V^{i,B}(x,t), \tag{6}$$

where $i = 1,2,\ A = 1,...,N,\ B = 1,...,M,$ hence, the averaged model of non-symmetric sandwich beam can be derived:

$$\begin{aligned}
&< A_E > \partial^3 W + < A_E \partial^3 h^A > Q^A + < B^i > \partial^2 U^i + < B^i \partial^2 g^{i,B} > V^{i,B}(x,t) + \\
&\qquad - < M > \partial \ddot{W} - < M \partial h^A > \ddot{Q}^A - < M^i > \ddot{U}^i - < M^2 g^{2,B} > \ddot{V}^{2,B} = 0, \\
&< A_E g^{1,K} > \partial^3 W + < A_E \partial^3 h^A g^{1,K} > Q^A + < B^i g^{1,K} > \partial^2 U^i + \\
&\qquad + < B^i \partial^2 g^{i,B} g^{1,K} > V^{i,B}(x,t) - < M g^{1,K} > \partial \ddot{W} - < M \partial h^A g^{1,K} > \ddot{Q}^A + \\
&\qquad - < M^2 g^{1,K} > \ddot{U}^2 - < M^i g^{i,B} g^{1,K} > \ddot{V}^{i,B} = 0, \\
&< C_E > \partial^3 W + < C_E \partial^3 h^A > Q^A + < \tilde{C} > \partial W + < \tilde{C} \partial h^A > Q^A + < D^i > \partial^2 U^i + \\
&\qquad + < D^i \partial^2 g^{i,B} > V^{i,B} + < G_c > (U^1 - U^2) + < G_c g^{1,B} > V^{1,B} + \\
&\qquad - < G_c g^{2,B} > V^{2,B} + < \tilde{M} > \partial \ddot{W} + < \tilde{M} \partial h^A > \ddot{Q}^A + \\
&\qquad - < \tilde{M}^i > \ddot{U}^i - < \tilde{M}^1 g^{1,B} > \ddot{V}^{1,B} = 0, \\
&< C_E g^{2,K} > \partial^3 W + < C_E \partial^3 h^A g^{2,K} > Q^A + < \tilde{C} g^{2,K} > \partial W + < \tilde{C} \partial h^A g^{2,K} > Q^A + \\
&\qquad + < D^i g^{2,K} > \partial^2 U^i + < D^i \partial^2 g^{i,B} g^{2,K} > V^{i,B} + < G_c g^{2,K} > (U^1 - U^2) + \\
&\qquad + < G_c g^{1,B} g^{2,K} > V^{1,B} - < G_c g^{2,B} g^{2,K} > V^{2,B} + < \tilde{M} g^{2,K} > \partial \ddot{W} + \\
&\qquad + < \tilde{M} \partial h^A g^{2,K} > \ddot{Q}^A - < \tilde{M}^1 g^{2,K} > \ddot{U}^1 - < \tilde{M}^i g^{i,B} g^{2,K} > \ddot{V}^{i,B} = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&< C_E > \partial^4 W + < C_E \partial^4 h^A > Q^A + < D^i > \partial^3 U^i + < D^i \partial^3 g^{i,B} > V^{i,B} + \\
&\quad + < \tilde{M} > \partial^2 \ddot{W} + < \tilde{M} \partial^2 h^A > \ddot{Q}^A - < \tilde{M}^i > \partial \ddot{U}^i - < \tilde{M}^i \partial g^{i,B} > \ddot{V}^{i,B} + \\
&\quad - < \mu > \ddot{W} = < -p_3 >, \\
&< C_E h^L > \partial^4 W + < C_E \partial^4 h^A h^L > Q^A + < D^i h^L > \partial^3 U^i + < D^i \partial^3 g^{i,B} h^L > V^{i,B} + \\
&\quad + < \tilde{M} h^L > \partial^2 \ddot{W} + < \tilde{M} \partial^2 h^A h^L > \ddot{Q}^A - < \tilde{M}^i h^L > \partial \ddot{U}^i + \\
&\quad - < \tilde{M}^i \partial g^{i,B} h^L > \ddot{V}^{i,B} - < \mu h^A h^L > \ddot{Q}^A = < -p_3 h^L >, \\
&< M^1 g^{1,B} >= 0, \qquad < \tilde{M}^2 g^{2,B} >= 0, \qquad < \mu h^A >= 0, \\
&i = 1,2, \qquad A, L = 1,2,...,N, \qquad B, K = 1,2,...,M.
\end{aligned} \tag{7cd.}$$

The system of equations (7) is a system of $3 + N + 2M$ partial differential equations with constant coefficients, which is relatively simple to solve. The basic unknown functions are macrodisplacements: $W(x,t), U^i(x,t)$, and fluctuation amplitudes: $Q^A(x,t), V^{i,B}(x,t)$, which quantity is dependent on the number of assumed fluctuation shape functions. System of equation (7) should be followed by four boundary conditions for $W(x,t)$, three boundary conditions for every each $U^i(x,t)$ and a two initial conditions for every each unknown function.

## 5. Calculation example

In this section, the free vibration analysis of a simply supported non-symmetric sandwich beam is performed. A basic periodicity cell of the considered structure is presented on Figure 3, while its geometry and material properties are given as follows:

$$\begin{aligned}
&E_1^{\,1} = 200\,\text{GPa}, \quad E_2^{\,1} = 100\,\text{GPa}, \quad E_1^{\,2} = 75\,\text{GPa}, \quad E_2^{\,2} = 30\,\text{GPa}, \quad E_c = 1\,\text{GPa}, \\
&\rho_1^{\,1} = 7850\,\tfrac{\text{kg}}{\text{m}^3}, \quad \rho_2^{\,1} = 4000\,\tfrac{\text{kg}}{\text{m}^3}, \quad \rho_1^{\,2} = 2710\,\tfrac{\text{kg}}{\text{m}^3}, \quad \rho_2^{\,2} = 1200\,\tfrac{\text{kg}}{\text{m}^3}, \quad \rho_c = 500\,\tfrac{\text{kg}}{\text{m}^3}, \\
&\nu_1^{\,1} = \nu_1^{\,2} = \nu_2^{\,1} = \nu_2^{\,2} = \nu_c = 0.34, \quad h_f^{\,1} = 2\text{mm}, \quad h_f^{\,2} = 1\text{mm}, \quad h_c = 80\text{mm}, \\
&b = 0.2\text{m}, \quad l = 0.2\text{m}, \quad L_1 = 5\text{m}, \xi \in < 0,1 >, \quad \zeta \in < 0,1 >.
\end{aligned} \tag{8}$$

The free vibration frequencies are evaluated with the use of equations (7) in two calculation cases, in which different sets of fluctuation shape functions are assumed:

$$\begin{aligned}
&\text{Case I:} && \text{Case II:} \\
&h^1 = l^4 \cos(2\pi x / l) + c_1, && h^1 = 0, \\
&g^{1,1} = l^3 \sin(2\pi x / l) + c_2, && g^{1,1} = l^3 \sin(2\pi x / l) + c_2, \\
&g^{2,1} = l^3 \sin(2\pi x / l) + c_3, && g^{2,1} = l^3 \sin(2\pi x / l) + c_3,
\end{aligned} \tag{9}$$

where constants $c_1, c_2, c_3$ are derived from normalizing conditions $(7)_{7\text{-}9}$. In both cases, the solutions to governing equations are assumed in the form, which satisfies the boundary conditions:

$$
\begin{aligned}
&W = A_W \sin(n\pi x / L_1)\sin(\omega t), && Q^1 = A_Q \sin(n\pi x / L_1)\sin(\omega t),\\
&U^1 = A_{U^1} \cos(n\pi x / L_1)\sin(\omega t), && V^{1,1} = A_{V^{1,1}} \sin(n\pi x / L_1)\sin(\omega t),\\
&U^2 = A_{U^2} \cos(n\pi x / L_1)\sin(\omega t), && V^{2,1} = A_{V^{2,1}} \sin(n\pi x / L_1)\sin(\omega t).
\end{aligned} \tag{10}
$$

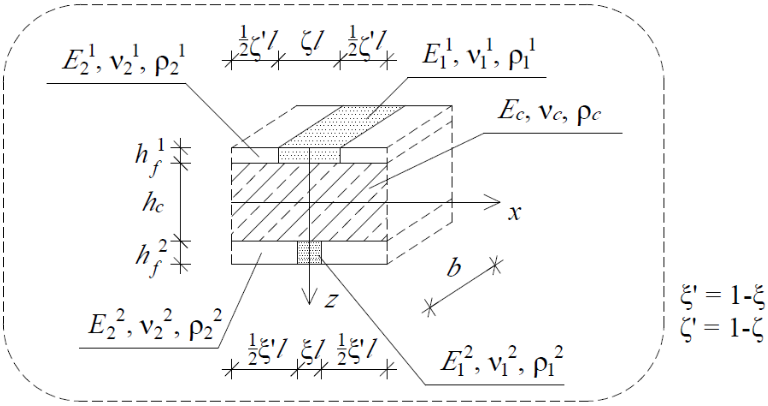


Figure 3. Basic periodicity cell of the considered non-symmetric sandwich beam.

The results of the obtained free vibration frequencies are compared with FEM model, in which the considered beam was modelled with the use of 2D elements with a proper dimensions, material properties and boundary conditions. The results of comparative studies are presented in Table 1.

Table 1. The free vibration frequencies [Hz] obtained within different calculation cases.

| Mode | $\xi = \zeta = 0.4$ | | | $\xi = 0.8$, $\zeta = 0.2$ | | |
|---|---|---|---|---|---|---|
| | Case I | Case II | FEM | Case I | Case II | FEM |
| 1 | 5.06 | 5.24 | 5.78 | 5.57 | 5.64 | 5.90 |
| 2 | 20.08 | 20.80 | 19.74 | 22.10 | 22.39 | 21.53 |
| 3 | 44.63 | 46.21 | 44.77 | 49.04 | 49.67 | 48.31 |
| 4 | 78.00 | 80.77 | 76.68 | 85.55 | 86.65 | 83.41 |
| 5 | 119.36 | 123.60 | 118.59 | 130.60 | 132.28 | 128.18 |
| 6 | 167.75 | 173.72 | 164.06 | 183.08 | 185.43 | 177.41 |

## 6. Conclusions

In this article the analysis of vibrations of periodic non-symmetric sandwich beam is performed with the use of the averaged model based on the tolerance averaging technique. The derived model is formulated with a system of governing equations with constant coefficients, which describes the vibrations of structure with complicated, periodic microstructure. Contrary to other methods of modelling, such as the asymptotic homogenization method, the proposed averaged model of periodic beam still allows us to investigate its microscale behaviour.

The described model can be considered to be very versatile, as it can be used in the analysis of the behaviour of a large variety of microheterogeneous beams with different types of inhomogeneities, such as: varying thickness and/or material properties of every each layer or even non-uniform distribution of the core. All of those cases can be investigated with the same calculation algorithm, which can be easily looped, in order to obtain a basic optimization process. Similar calculations with the use of alternative modelling methods, such as FEM, is a time-consuming process, which additionally requires a lot of computational resources, due to a highly refined mesh. In such case, the superiority of the proposed averaged model can be easily seen.

In the calculation example, the proof of the correctness of the modelling procedure is presented, as the free vibration frequencies obtained within the averaged model are compared with the results of FEM analysis. By analysing the results of calculations presented in Table 1, one can perceive a satisfying consistency of results. It can be observed, that the free vibration frequencies of the higher modes in Case II are characterised by a slightly higher relative error (when compared to FEM) than the results of Case I. On the other hand, the governing equations of Case II contain less terms than similar equations in Case I, hence, they can be considered more convenient in use.

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