Mathematical Model and Method of Determination of Amplitude-Frequency Characteristics of Corrugated Cylindrical Shells for Geometrically Nonlinear Vibrations

Mykhailo MARCHUK
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU
3 b Naukova Str., L'viv, 79053, Ukraine
and National University «L'viv Polytechnic»
12 Bandera Str., L'viv, 79013, Ukraine
mv_marchuk@ukr.net; mvmarchuk56@gmail.com

Taras GORIACHKO
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU
3 b Naukova Str., L'viv, 79053, Ukraine
taras.horyachko@gmail.com

Bohdan KINDRATSKYY
National University «L'viv Polytechnic»
12 Bandera Str., L'viv, 79013, Ukraine
me@in.lviv.ua

Vira PAKOSH
Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU
3 b Naukova Str., L'viv, 79053, Ukraine
v.pakosh@ukr.net

Abstract

On the basis of spatial geometrically nonlinear theory of elasticity by using quadratic approximations for components of the vector of displacements along the normal coordinate to the median surface is developed a new mathematical model of nonlinear dynamics of cylindrical shells with corrugated configuration for median surface. The perturbation method for the solution of systems of nonlinear differential equations for problems of determination of amplitude-frequency characteristics is generalized. In a combination of finite element and generalized perturbations methods, a new methodology for solving problems of free geometrically nonlinear vibrations of shells with complex guide geometry was developed and verified. With its help the influence of geometrical parameters of corrugating at circular coordinate on the main frequency of the elongated cylindrical panel is investigated.

***Keywords**:* corrugated shell, geometrically nonlinear vibrations, mathematical model, generalized method of perturbations, amplitude-frequency characteristics

1. Introduction

Cylindrical shells are one of the most common elements of loaded constructions, mechanisms and devices for various purposes. This is due to their rational material consumption and the ability to provide the necessary rigidity in certain directions, which are determined by the peculiarities of operating conditions. As a rule, the stiffness in a given direction of smooth shells is regulated by selecting the elastic characteristics of the used material during manufacture. However, in this way it is not always possible to achieve the required value of stiffness in this direction. Therefore, in combination with the above approach, methods are used to complicate the geometry of the middle surfaces of the shells [3]. For cylindrical shells, this approach is implemented by corrugating of the guide curve and/or generatrix [4, 5, 8]. The main property of such shells is their high anisotropy and stiffness in the transverse to the corrugating direction. We will note that the specified covers owing to insurmountable at present mathematical difficulties are investigated insufficiently. This is especially true of geometrically nonlinear deformation. In this direction it is necessary to note works [6, 7, 16, 17] in which questions of linear and geometrically nonlinear vibrations of corrugated cylindrical shells are considered.

A significant number of works in which analytical and numerical solutions is given are devoted to the study of geometrically nonlinear vibrations of cylindrical shells on the basis of classical and generalized models of deformation. References to them can be found in [1, 18]. At the same time, these models do not take into account the spatial stress-strain state, in particular the pliability to transverse compression. This property is inherent in most modern materials, especially for polymer-based composites. Also, most of them neglect the components of the elasticity tensor corresponding to the transverse deformations. This can lead to significant errors in the design calculations of real thin-walled structures both in terms of strength and the elimination of resonant phenomena under the action of cyclic loads. Therefore, the need to improve existing and develop new mathematical models and methods for calculating the dynamic geometrically nonlinear deformation of cylindrical shells with complex geometry of the middle surface, including corrugated, is fully motivated. This allows you to create an effective methodology for solving problems of determining their amplitude-frequency characteristics.

2. 3-D formulation of the problem

Let's consider the curvilinear elastic layer attributed to the cylindrical coordinate system $\alpha_1 = \varphi$, $\alpha_2 = z$, $\alpha_3 = r$ of thickness h. Its equations of vibrations for geometrically nonlinear deformation have the form [7]:

$$div\hat{S} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}, \qquad (1)$$

where $\hat{S}$ – nonsymmetrical Kirchhoff stress tensor whose components are defined as follows:

$$S^{ij} = \sum_k \sigma^{ik} (\delta_k^j + \nabla_k u^j),$$

$\vec{u} = u_i \vec{e}_i \vec{e}_j$ – displacement vector, σ^{ik} – components of a symmetric Piola stress tensor ($\hat{\Sigma}$). The components of the strain tensor $\hat{E}$ will be determined by the components of the displacement vector according to the following relations

$$\varepsilon_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i + \nabla_i u^k \nabla_j u_k).$$

Suppose that the generalized Hooke's law is implemented for the layer of anisotropic elastic body. Then we obtain the following relationship between the components of the strain and stress tensors (elasticity relations)

$$\hat{\Sigma} = \tilde{C} \otimes \hat{E} . \tag{2}$$

In equations (1) and (2) $\tilde{C}$ – tensor of elastic characteristics of the layer, and ρ – its density; t – time coordinate.

Boundary conditions on the front surfaces of the shell $\alpha_3 = \pm h/2$ for the free vibrations has the form

$$S^{3i}(\alpha_1, \alpha_2, \pm h/2, t) = 0 , \quad |\alpha_i| \le \alpha_i^0, \ i = 1, 2 , \tag{3}$$

and on its ends $\alpha_1 = \pm\alpha_1^0$ at their hinged fixing on the lower front surface $\alpha_3 = -h/2$:

$$S^{1i}(\pm\alpha_1^0, \ \alpha_2, \ \alpha_3, \ t) = 0 , \tag{4}$$

$$u_i(\pm\alpha_1^0, \ \alpha_2, -h/2, \ t) = 0 , \quad i = 1, 2, 3 . \tag{5}$$

To solve the current problem, we considered the initial conditions in the form:

$$u_i(\alpha_1, \alpha_2, \alpha_3, t)\Big|_{t=t_0} = v_i^0(\alpha_1, \alpha_2, \alpha_3, t_0) ,$$

$$\left.\frac{\partial u_i(\alpha_1, \alpha_2, \alpha_3, t)}{\partial t}\right|_{t=t_0} = v_i^1(\alpha_1, \alpha_2, \alpha_3, t_0) . \tag{6}$$

Here we have v_i^k $(k = 0, 1; \ i = 1, 2, 3)$ – predefined functions.

The motion equations (1) together with relations (2), boundary (3)–(5) and initial (6) conditions are a three-dimensional mathematical model in differential form that describes geometrically nonlinear vibrations of an elastic curvilinear layer.

3. Variation problem statement

In the general case, this problem is not amenable to analytical solution due to the currently insurmountable mathematical difficulties. Therefore, in most cases, when considering real structures using numerical methods. One of the most effective numerical methods for solving the above class of problems is the finite element method [2]. The vast majority of variants of this method are based on equivalent variation formulations of problems.

For the variation formulation of the problem in this case, the principle of virtual work is used, which states that the work of internal forces is equal to the work of external forces on any virtual displacements. In integral form, this statement is written as follows [8]:

$$\int_{V(t)} \delta\hat{E} : \hat{\tau}\, dV + \int_{V(t)} \rho\delta\vec{u} \cdot \frac{\partial^2 \vec{u}}{\partial t^2} dV = 0 , \tag{7}$$

$$\forall \delta\vec{u} \in D_A = \{\vec{u} : \vec{u} \in W_2^{(1)}; \ \ \vec{u} = \vec{v}(\alpha_1, \alpha_2, \alpha_3) \in \partial V_u, \forall t\},$$

where V – the volume occupied by the elastic layer; $W_2^{(1)}$ – Sobolev's space; $\delta\hat{E}$ – the linear strain tensor, which corresponds to the variation of displacements $\delta\vec{u}$; $\hat{\tau}$ – Cauchy stress tensor.

Since $V(t)$ is unknown in the case of geometrically nonlinear deformations, in which the volume $V(t)$ differs significantly from the initial volume V_0, (7) in the initial (undeformed) configuration has the form:

$$\int_{V_0} \delta \overset{0}{\hat{\varepsilon}} : \overset{0}{\hat{S}}\, dV + \int_{V_0} \rho_0\, \delta \overset{0}{\vec{u}} \cdot \frac{\partial^2 \overset{0}{\vec{u}}}{\partial t^2}\, dV = 0, \tag{8}$$

where $\overset{0}{\hat{S}}$ is the second symmetric Piola-Kirchhoff stress tensor, for which the formula is valid $\hat{P} = \hat{S} \cdot (\hat{F}^{-1})^T$, where $\hat{S}$ is the tensor from formula (1), and $\hat{F} = \hat{G} + grad^{\,T} u$ is the local motion gradient tensor; $\delta \overset{0}{\hat{\varepsilon}}$ – Green's deformation tensor, which corresponds to the variation of displacements $\delta \overset{0}{\vec{u}}$.

"0" above the tensor and vector symbols means that their components are considered in the initial undeformed coordinate system.

4. Geometry of an elongated cylindrical corrugated shell

We have this case when $h/L << 1$ and

$$V_0 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 \in [0, L],\ \alpha_2 \in [-\infty, +\infty],\ \alpha_3 \in [-h/2, h/2]\},$$

where the coordinate system (α_1 , α_2 , α_3) corresponds to the corrugated middle surface [5], L − the length of the arc of guide.

The relationships between the Cartesian (x_1 , x_2 , x_3) and local coordinate systems are as follows

$$x_1 = (qR + g_A \cos(g_v \theta)) \cos(\theta); \quad x_2 = \alpha_2; \\ x_3 = (qR + g_A \cos(g_v \theta)) \sin(\theta). \tag{9}$$

In (9) $\theta = \theta(\alpha_1) = \dfrac{\pi}{2} + \dfrac{1}{R}\left(\dfrac{L}{2} - \alpha_1\right)$; $q = q(\alpha_3) = 1 + \alpha_3 / R$; g_A – amplitude of corrugation (Fig. 1a); g_v − its frequency; R − the distance from the axis coordinate to the middle surface of the non-corrugated cylindrical layer (Fig. 1b).

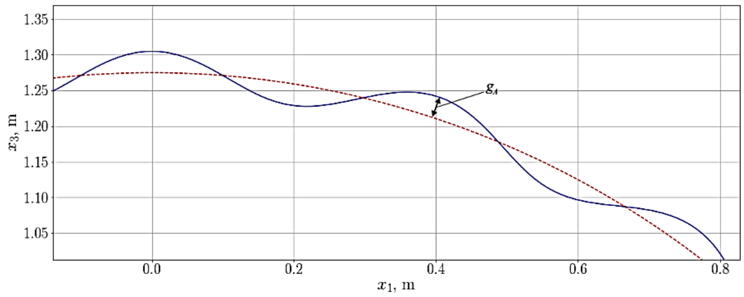

Figure 1a. Illustration of the geometric content of the corrugation amplitude g_A

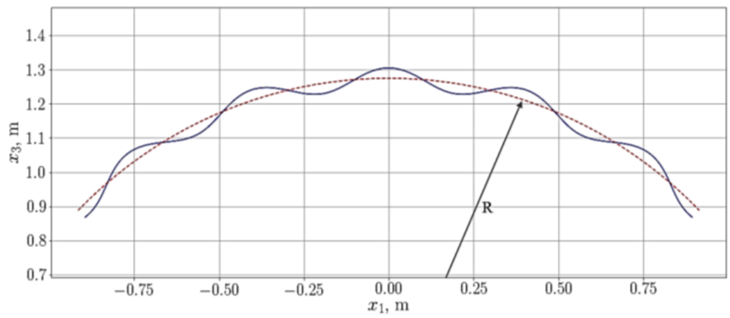

Figure 1b. Location of the middle surface of the corrugated layer relative to the basic cylindrical surface of the radius R

Using (9) for the radius-vector of the middle surface of the corrugated layer we obtain

$$\vec{r}(\alpha_1) = (R + g_A Cos\,(g_v\theta))\,Cos\,(\theta)\vec{e}_1 + \vec{e}_2 + (R + g_A Cos\,(g_v\theta))\,Sin\,(\theta)\vec{e}_3\,, \qquad (10)$$

where $\vec{e}_i$, $i = 1,2,3$ are the base vectors of the Cartesian coordinate system.

The expression tangent to the middle surface can be given the form

$$\vec{r}\,'(\alpha_1) = \frac{\partial \vec{r}}{\partial \alpha_1} = (w\,Sin(\theta) + z\,Cos(\theta))\,\vec{e}_1 + \vec{e}_2 + (-w\,Cos(\theta) + z\,Sin(\theta))\,\vec{e}_3\,, \qquad (11)$$

where

$$w = 1 + \frac{g_A}{R} Cos(g_v\theta)\,;\; z = \frac{g_A g_v}{R} Sin(g_v\theta)\,. \qquad (12)$$

Taking into account (11), the expression for the normal to the middle surface has the form

$$\vec{n}(\alpha_1) = \frac{w Cos(\theta) - z Sin(\theta)}{\sqrt{w^2 + z^2}} \vec{e}_1 + \vec{e}_2 + \frac{w Sin(\theta) + z Cos(\theta)}{\sqrt{w^2 + z^2}} \vec{e}_3, \tag{13}$$

Given (13), and the fact that $\vec{n}(\alpha_1) \cdot \vec{n}(\alpha_1) = 1$ and $\frac{\partial \vec{n}}{\partial \alpha_1} \cdot \frac{\partial \vec{n}}{\partial \alpha_1} = 0$, for the covariant components of the metric tensor we obtained

$$g_{11} = \frac{\partial \vec{r}}{\partial \alpha_1} \cdot \frac{\partial \vec{r}}{\partial \alpha_1}; \; g_{22} = 1; \; g_{33} = 1; \; g_{ij} = 0, \; i \neq j; \, i, j = 1,2,3.$$

The Lame's parameters for the middle surface of the corrugated layer, which are necessary for determining the components of the tensors in equation (8), are determined by the formulas

$$H_1^2 = \frac{\partial \vec{r}}{\partial \alpha_1} \cdot \frac{\partial \vec{r}}{\partial \alpha_1}; \; H_2^2 = 1; \; H_3^2 = 1. \tag{14}$$

Therefore, for an elongated cylindrical corrugated shell we have

$$H_1 = A(\alpha_1)(1 + K(\alpha_1)\alpha_3), \quad H_2 = 1, \quad H_3 = 1, \tag{15}$$

where $A(\alpha_1)$ – the coefficient of the first quadratic shape, and $K(\alpha_1)$ – the main curvature in the direction the coordinates α_1 of the middle corrugated surface [5]:

$$A(\alpha_1) = \sqrt{w^2 + z^2}, \qquad K(\alpha_1) = (wy + \frac{2}{R} z^2)/(w^2 + z^2)^{\frac{3}{2}}.$$

Here the values of w and z are defined in (12); $y = -\frac{1}{R} w + \frac{g_A g_v^2}{R^2} cos\,(g_v \theta)$.

5. Reduction of Two-Dimensional Problem to One-Dimensional One

If the thickness h of the layer is much less than the length of the arc of the cross section $\alpha_2 = 0$ of the middle surface $\alpha_3 = 0$, we can consider the components of the displacement vector u_1, u_3 distributed by the quadratic law along the coordinate α_3 [6]:

$$\begin{aligned} u_1 &= u_1(\alpha_1, \alpha_3, t) = u_{10}(\alpha_1, t), P_0(\alpha_1) + u_{11}(\alpha_1, t), P_1(\alpha_3) + u_{12}(\alpha_1, t), P_2(\alpha_3); \\ u_3 &= u_3(\alpha_1, \alpha_3, t) = u_{30}(\alpha_1, t), P_0(\alpha_3) + u_{31}(\alpha_1, t), P_1(\alpha_1) + u_{32}(\alpha_1, t), P_2(\alpha_3), \end{aligned} \tag{16}$$

where polynomials

$$P_0(\alpha_1) = 1/2 - \alpha_3 / h; \; P_1(\alpha_1) = 1/2 + \alpha_3 / h; \; P_2(\alpha_1) = 1 - 2(\alpha_1 / h)^2, \tag{17}$$

and the coefficients $u_{ij}(\alpha, t), \; i = 1, 3; \; j = 0, 1, 2$ are unknown.

Substitution (16) in the strain relations allows to obtain expressions for the components of the strain tensor $\hat{E}$, which depend on only one coordinate α_1. Similarly, we obtain expressions for the components of the stress tensor $\hat{\Sigma}$. Then after using the specified expressions in (8) and integration by the variable α_3 we obtain [5, 6]:

$$\int_0^L \vec{\varepsilon}\, C \cdot \vec{\varepsilon} \delta u A(\alpha_1)\, d\alpha_1 + \int_0^L \rho\, B \cdot \frac{\partial^2 u}{\partial t^2} A(\alpha_1)\, d\alpha_1 = 0, \tag{18}$$

where

$$\vec{\varepsilon}\,' = (\varepsilon_{110}, \varepsilon_{111}, \varepsilon_{112}, \varepsilon_{330}, \varepsilon_{331}, \varepsilon_{332}, \varepsilon_{130}, \varepsilon_{131}, \varepsilon_{132})^T ;$$

$$\vec{u}\,' = \left(u_{10}, \frac{du_{10}}{d\alpha_1}, u_{11}, \frac{du_{11}}{d\alpha_1}, u_{12}, \frac{du_{12}}{d\alpha_1}, u_{30}, \frac{du_{30}}{d\alpha_1}, u_{31}, \frac{du_{31}}{d\alpha_1}, u_{32}, \frac{du_{32}}{d\alpha_1} \right)^T ;$$

C' – block matrix of size 9x9:

$$C' = \begin{pmatrix} C_{11}^1 & C_{13}^1 & 0 \\ C_{13}^1 & C_{133}^1 & 0 \\ 0 & 0 & C_{55}^1 \end{pmatrix}, \quad C'_{ij} = C'_{ij} h \begin{pmatrix} 1/3 & 1/6 & 1/3 \\ 1/6 & 1/3 & 1/3 \\ 1/3 & 1/3 & 8/15 \end{pmatrix},$$

C'_{ij} – elements of the matrix of elastic characteristics of the orthotropic material for the layer at $i, j = 1,3,5$, when recorded through the components of the tensor $\hat{E}$ (2);

B' – block matrix size 12x12:

$$B' = \begin{pmatrix} B_0 & 0 \\ 0 & B_0 \end{pmatrix},$$

and the matrix B_0 of dimension 6x6 has the form

$$B_0 = h(b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}),$$

where

$$b_{11} = (1/3;\ 0;\ 1/6;\ 0;\ 1/3;\ 0)^T ; \quad b_{13} = (1/6;\ 0;\ 1/3;\ 0;\ 1/3;\ 0)^T ;$$

$$b_{15} = (1/3;\ 0;\ 1/3;\ 0;\ 8/15;\ 0)^T ; \quad b_{12} = b_{14} = b_{16} = (0;\ 0;\ 0;\ 0;\ 0;\ 0)^T .$$

Thus, the equation (18) is a one-dimensional variation problem on free vibrations of an elongated corrugated cylindrical panel constructed using quadratic approximations of displacements at the normal coordinate to the middle surface of the considered thin orthotropic elastic layer.

6. Solution Method

To find the solution of the variation problem (18), we first use a one-dimensional scheme of the finite element method with linear approximations of unknown coefficients $u_{ij}(\alpha_1, t)$ by the coordinate α_1 [2]. This allowed to obtain the resulting equation

$$K_L\vec{U} + [K_{NL}^{(1)}(\vec{U}) + K_{NL}^{(2)}(\vec{U},\vec{U})]\vec{U} + M\ddot{\vec{U}} = 0, \tag{19}$$

here $\vec{U}$ is the vector of all unknown nodal values of the coefficients $u_{ij}^k(t)$; K_L – linear, and $K_{NL}^{(1)}$, $K_{NL}^{(2)}$ – nonlinear stiffness matrices; M – matrix of masses [12].

The solution of the system of nonlinear ordinary differential equations (19) is found by the perturbation method proposed in [10, 11] and generalized by the authors [12, 13]. Its essence is to consider a perturbed system of equations

$$K_L\vec{U} + \mu[K_{NL}^{(1)}(\vec{U}) + K_{NL}^{(2)}(\vec{U},\vec{U})]\vec{U} + M\ddot{\vec{U}} = 0, \tag{20}$$

where $\mu\,(0 \le \mu \le 1)$ is the perturbation parameter.

In the classical case, the solution of the system (20) is given in the form

$$\vec{U}(t) = \vec{U}_0(t) + \mu\vec{U}_1(t) + \mu^2\vec{U}_2(t) + O(\mu^3). \tag{21}$$

However, with such a representation we get in the solution a secular term that is proportional to $t\sin(\omega t)$. Therefore, it was proposed to generalize the perturbation method for the solving of the system of equations (20). This generalization consists in representing the linear stiffness matrix in the form

$$K_L = K - \mu K_{L1} - \mu^2 K_{L2} + O(\mu^3). \tag{22}$$

The consequence of substitution (21) and (22) in (20) and grouping of expressions at equal degrees μ are the following equations:

$$K\vec{U}_0(t) + M\ddot{\vec{U}}_0(t) = 0,$$

$$K\vec{U}_1(t) + M\ddot{\vec{U}}_1(t) =$$
$$= K_{L1}\vec{U}_0(t) - K_{NL}^1(\vec{U}_0(t))\vec{U}_0(t) - K_{NL}^2(\vec{U}_0(t),\vec{U}_0(t))\vec{U}_0(t);$$

$$K\vec{U}_2(t) + M\ddot{\vec{U}}_2(t) =$$
$$= K_{L1}\vec{U}_1(t) + K_{L2}\vec{U}_0(t) - K_{NL}^1(\vec{U}_0(t))\vec{U}_1(t) - K_{NL}^1(\overleftarrow{U}_1(t))\vec{U}_0(t) -$$
$$- K_{NL}^2(\vec{U}_0(t),\vec{U}_0(t))\vec{U}_1(t) - 2K_{NL}^2(\vec{U}_0(t),\vec{U}_1(t))\vec{U}_0(t). \tag{23}$$

The system (23) is linear with respect to the desired components $\vec{U}_0(t)$, $\vec{U}_1(t)$ and $\vec{U}_2(t)$ of approximation of the vector of nodal displacements $\vec{U}(t)$.

Currently, there are a large number of methods for approximate analytical and numerical integration of such systems of ordinary differential equations. The authors in [12] proposed a method for obtaining a finite number of the first eigenvalues of the system (23) and forms of vibrations. Its verification was performed by comparison with the results of [9].

7. Numerical Results

A corrugated in circular direction cylindrical shell with a guide length $L = 2$ m, the radius of the middle surface of the shell, the front surfaces of which intersect the tops of the corrugations, $R = 1.25$ m, its thickness $h = 0.05$ m, elastic characteristics: $E_1 = 2.1 \cdot 10^{11}$ N/m^2, $\nu_{13} = 0.3$, $G_{13} = 8.1 \cdot 10^{10}$ N/m^2 and density $\rho = 8 \cdot 10^3$ kg/m^3 is considered.

To study the influence of corrugation parameters on the amplitude-frequency characteristics of free vibrations at geometrically nonlinear deformation of the specified shell, we use the introduction in [18] of the concept of skeletal curves.

They graphically illustrate the interdependence of dimensionless main own frequency ω_{NL} / ω_L and dimensionless amplitude $w_{\max} / h$. Here ω_{NL} and $w_{\max}$ – the main own frequency and the corresponding amplitude for geometrically nonlinear vibrations, and ω_L – the main own frequency for linear vibrations.

In fig. 2 the skeletal curves are given, which show the relationship between the dimensionless main own frequency and the amplitude of geometrically nonlinear radial vibrations of the considered shell at different values of the corrugation frequency g_v.

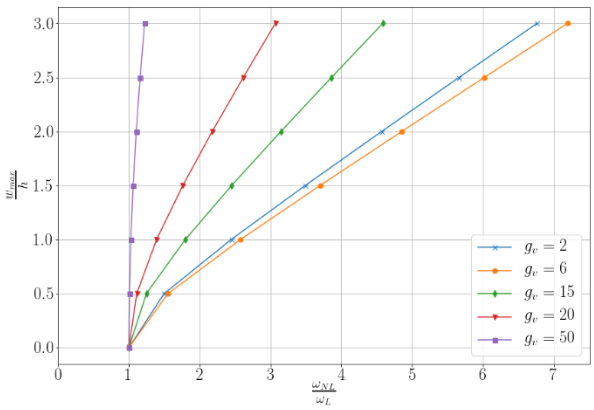

Figure 2. Dependence between dimensionless main own frequency and amplitude of nonlinear vibrations at different values of corrugation frequency.

Since the stiffness of the shell [18] is proportional to the main own frequency of its free vibrations, in this case the maximum stiffness is achieved at the corrugation frequency $g_v = 6$.

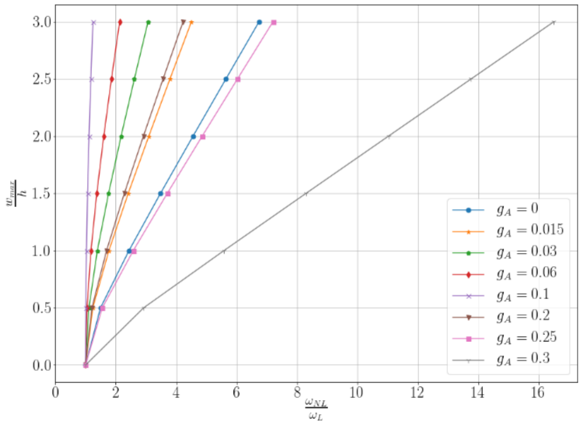

Figure 3. Dependence between dimensionless main own frequency and amplitude of nonlinear vibrations at different values of corrugation amplitude g_A.

In fig. 3 shows the skeletal curves of the relationship between the dimensionless main own frequency and the amplitude of geometrically nonlinear radial vibrations of the shell at different values of the corrugation amplitude. The values at which the stiffness of the corrugated shell in the transverse direction is less than the stiffness of the non-corrugated shell are revealed.

8. Conclusions

A methodology for determining the amplitude-frequency characteristics of elongated corrugated cylindrical shells under geometrically nonlinear deformation has been developed. It consists of a constructed mathematical model of the process of nonlinear vibrations and an analytical-numerical method of its implementation. Using it, the influence of corrugation parameters on the main own frequency of nonlinear vibrations of the specified type of shells is analyzed. This revealed the presence of such values of the parameters of corrugation, which achieves the maximum value of the stiffness of a particular elongated cylindrical shell. This result can be the basis for formulating the problem of the body of the optimal values of the parameters of corrugation in the circular direction of the elongated cylindrical shells according to the criteria of maximum rigidity with minimum material consumption.

References

1. M. Amabili, *Nonlinear vibrations and stability of shells and plates*, Cambridge Univ. Press, Cambridge 2008.
2. K.-J.Bathe, *Finite Element Procedures*, 2nd Ed., Watertown, MA 2014.
3. I. Dayyani, A. D. Shaw, E. I. Saavedra Flores, M. I. Friswell, *The Mechanics of Composite Corrugated Structures: A Review with Applications in Morphing Aircraft*, Composite Structures, 133 (2015) 358-380
4. T. G. Ghazijahani, H. Jiao, D. Holloway, *Longitudinally stiffened corrugated cylindrical shells under uniform external pressure*, Journal of Constructional Steel Research, 110 (2015) 191–199.
5. G.R. Gulgazaryan, L.G. Gulgazaryan, *Vibrations of a corrugated orthotropic cylindrical shell with free edges*, Int. Appl. Mech., 42, Issue 12 (2006) 1398–1413.
6. Z. He, G. Liu, W. Jiang, Z. Huang, D. Zhang, *Nonlinear Dynamic Responses of a Corrugated Shell Structure under Uniform Load*, J. of Eng. Mech., 140, Issue 6 (2014)
7. Y.-W. Kim, *Vibration Analysis of Longitudinally Corrugated Cylindrical Shells*, Transactions of the Korean Soc. for Noise and Vibration Eng., 26 (2016) 851– 856.
8. A. M. Khalifa, *Solving the Vibration Problem of Inhomogeneous Orthotropic Cylindrical Shells with Hoop-Corrugated Oval Cross Section*, Comptes Rendus Mécanique, 343, No.9 (2015) 482–494
9. L.V. Kurpa, N.A. Budnikov, *Investigation of forced nonlinear vibrations of multilayer hollow shells by means of multimode approximation*, Bulletin of Donetsk National University. Series A. Natural Sciences, No.1 (2013) 55–60 [in Russian].
10. R. Lewandowski, *Analysis of strongly nonlinear free vibration of beams using perturbation method*, Civil and Env. Eng. Reports, 1 (2005) 153–168.
11. R. Lewandowski, *Free Vibration of Structures with Cubic Non-linearity-remarks on Amplitude Equation and Rayleigh Quotient*, Comput. Methods Appl. Mech. Eng., 192(13) (2003) 1681–1709.
12. M. Marchuk, T. Goriachko, V. Pakosh, *Geometrically Nonlinear Free Transversal Vibrations of Thin-Walled Elongated Panels with Arbitrary Generatrix*, Vibrations in Physical Systems, 26 (2014) 153–160.
13. M. Marchuk, T. Goriachko, V. Pakosh, *Natural Frequencies of Layered Elongated Cylindrical Panels for Geometrically Nonlinear Deformation at Discrete Consideration of Components*, Vibrations in Physical Systems, 27 (2016) 255–264.
14. N. F. Morozov, *Selected two-dimensional problem of elasticity theory*, Publishing House of Leningrad. University Press, Leningrad 1978.
15. J. N. Reddy, *An introduction to nonlinear finite element analysis*, Oxford Univ. Press, 2004.
16. N. P. Semenyuk, I. Yu. Babich, N. B. Zhukova, *Natural Vibrations of Corrugated Cylindrical Shells*, Int. Appl. Mech., 41, No.5 (2005), 512–519.
17. Werner Soedel, *Vibrations of Shells and Plates, 3rd Ed., Rev. and Expanded*, Marcel Dekker, Inc., NY 2004.
18. A. S. Vol'mir, *Nonlinear dynamics of plates and shells*, Nauka, Moscow 1972 [in Russian].