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Free Vibrations of Microstructured Functionally Graded Plate Band with Clamped Edges

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Abstract In this paper there are presented free vibrations of thin functionally graded plate band. This kind of plates has tolerance-periodic microstructure on the microlevel in planes parallel to the plate midplane. Dynamic problems of plates of this kind are described by partial differential equations with highly oscillating, tolerance-periodic, non-continuous coefficients. Thus, there are proposed here two models describing these plates by equations with smooth, slowly-varying coefficients. As an example there are analyses of free vibration frequencies for thin functionally graded plate band clamped on both edges. Using the known Ritz method the frequencies are obtained in the framework of proposed two models – the tolerance model and the asymptotic model.

Keywords: thin functionally graded plate band, microstructure, free vibrations, tolerance modelling

1. Introduction

In this paper, free vibrations of thin functionally graded plate band with span L along the x_1-axis are considered. These plate bands have the functionally graded structure on the macrolevel and on the microlevel their structure is tolerance-periodic in x_1. The size of microstructure is determined by l (length of the cell), being very small compared to the plate span L. A fragment of the plate band is shown in Fig. 1.

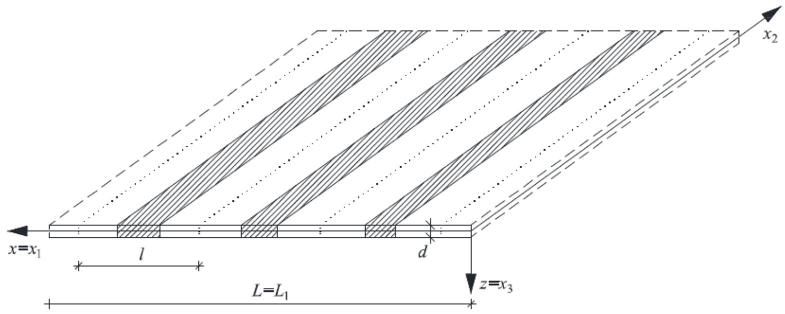

Fig. 1. A fragment of a thin functionally graded plate band

One of the most frequently method used to describe the functionally graded structure is averaging approaches for macroscopically homogeneous structure, e.g. periodic. Some of these methods are presented in [1]. Between them it can be distinguished models based on the asymptotic homogenization, e.g. applied for thin periodic plates [2]. Various analytical and numerical models were also proposed and used for microheterogeneous structures, e.g.: for sandwich beams with variable properties of core [3]; for laminated composite plates [4, 5]; for composite bars with helical distribution of constituents [6]. Unfortunately, the effect of the microstructure size is neglected in the governing equations of them.

The effect of the microstructure size can be taken into account using the tolerance averaging technique [7, 8]. This method makes it possible to investigate various dynamical, stability and thermoelastic problems for periodic structures, e.g.: vibrations of medium thickness plates [9]; static problems of thin plates with moderately large deflections [10]; vibrations of thin periodic plates resting on an elastic periodic foundation [11]; stability of thin cylindrical shells [12]; the geometrically nonlinear dynamics of periodic beams [13].

The tolerance modelling can be also applied to consider various problems of thermomechanics for functionally graded structures, e.g. for dynamic and stability problems of thin transversally graded plates with microstructure size bigger than the plate thickness [14, 15, 16]; for thin functionally graded plates with the microstructure size of an order of the plate thickness [17]; for heat distribution of composite cylindrical conductors having non-uniform microstructure [18]; for problems of thermoelasticity in transversally graded laminates [19]; dynamics of thin functionally graded microstructured shells [20-22].

The aim of this work is to apply the tolerance model and the asymptotic model to present the free vibration frequencies of thin plate band with tolerance-periodic microstructure in planes parallel to the midplane, which is clamped on the both edges.

2. Equations

Let us denote $x = x_1$, $z = x_3$, $x \in [0, L]$, $z \in [-d/2, d/2]$, where d is a constant plate thickness. It is assumed that the considerations are independent of the x_2-coordinate. The plate band is described by Ω, Ω=(0, L), with "the basic cell" $\Delta \equiv [-l/2, l/2]$ in the interval Ω, where l is the cell length, satisfying conditions $d << l << L$. The plate band is made of two elastic, isotropic materials, perfectly bonded across interfaces. The materials are characterised by Poisson's ratios ν', ν'', Young's moduli E', E'' and mass densities ρ', ρ''. The plate material structure can be treated as functionally graded in the x-axis direction if it is assumed: $E' \neq E''$ and $\rho' \neq \rho''$. The plate band properties – the mass density $\mu(x)$, the rotational inertia $\vartheta(x)$ and the bending stiffness $B(x)$ – are described by tolerance-periodic functions in x

$$\mu(x) \equiv d\rho(x) \quad \vartheta(x) \equiv \frac{d^3}{12}\rho(x) \quad B(x) \equiv \frac{d^3}{12(1-\nu^2)}E(x). \tag{1}$$

Let ∂ denote the derivative with respect to x, and $w(x, t)$ be deflection of the plate band ($x \in \Omega$, $t \in (t_0, t_1)$). Using the Kirchhoff plate theory, free vibrations of thin functionally graded plate band can be described by the partial differential equation of the fourth order with respect to the deflection w

$$\partial\partial(B\partial\partial w) + \mu\ddot{w} - \partial(\vartheta\partial\ddot{w}) = 0 \tag{2}$$

with highly-oscillating, non-continuous, tolerance-periodic coefficients.

Following the books by [8, 23] the fundamental modelling assumptions can be formulated. The first assumption is the micro-macro decomposition in which the deflection w appears in the form

$$w(x,t) = W(x,t) + h^A(x)V^A(x,t) \qquad A = 1, \dots, N, \qquad x \in \Omega, \tag{3}$$

where $W(\cdot, t)$, $V^A(\cdot, t) \in SV^2_\xi(\Omega, \Delta)$ (for every t) are the basic kinematic unknows called the macrodeflection and the fluctuation amplitudes, respectively, being slowly-varying functions in x (cf. [8, 23]); and the known fluctuation shape functions $h^A(\cdot) \in FS^2_\xi(\Omega, \Delta)$.

The second modelling assumption is the tolerance averaging approximation, where the terms of an order of $O(\xi)$ are treated as negligibly small in the course of modelling.

The modelling procedure of tolerance technique was shown in [8, 23]. The first step of this procedure is the formulation of the action functional

$$\mathcal{A}(w(\cdot)) = \int_\Omega \int_{t_0}^{t_1} \mathcal{L}(y, \partial\partial w(y,t), \partial w(y,t), \dot{w}(y,t), w(y,t))dydt\,, \tag{4}$$

where the lagrangean $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}(\mu\dot{w}\dot{w} + \vartheta\partial\dot{w}\partial\dot{w} - B\partial\partial w\partial\partial w). \tag{5}$$

The next step is substituting micro-macro decomposition (3) into lagrangean (5). Using the tolerance averaging approximation, the tolerance averaged lagrangean is obtained in form

$$\begin{aligned}\langle \mathcal{L}_h \rangle = -\frac{1}{2}\{(\langle B\rangle\partial\partial W + 2\langle B\partial\partial h^B\rangle V^B)\partial\partial W + \langle\vartheta\rangle\partial\dot{W}\partial\dot{W} + \langle B\partial\partial h^A\partial\partial h^B\rangle V^A V^B - \langle\mu\rangle\dot{W}\dot{W} + \\ + (\langle\vartheta\partial h^A\partial h^B\rangle - \langle\mu h^A h^B\rangle)\dot{V}^A\dot{V}^B\},\end{aligned} \tag{6}$$

where the macrodeflection W and the fluctuation amplitudes V^A, A = 1, …, N, are new basic kinematic unknows. The known fluctuation shape functions h^A are introduced in micro-macro decomposition (3).

From the principle of stationary action applied to the averaged functional $\mathcal{A}_h$ with lagrangean (6), after some mathematical manipulations, the following system of equations for W and V^A is obtained

$$\begin{gathered}\partial\partial(\langle B\rangle\partial\partial W+\langle B\partial\partial h^B\rangle V^B)+\langle\mu\rangle\ddot{W}-\langle\vartheta\rangle\partial\partial\ddot{W}=0\\ \langle B\partial\partial h^A\rangle\partial\partial W+\langle B\partial\partial h^A\partial\partial h^B\rangle V^B+\left(\underline{\langle\mu h^A h^B\rangle}+\underline{\langle\vartheta\partial h^A\partial h^B\rangle}\right)\ddot{V}^B=0.\end{gathered}\tag{7}$$

The Equations (7) are system of N+1 differential equations of the tolerance model of thin functionally graded plate bands. The coefficients of these equations are slowly-varying functions in *x*. The underlined terms in Equations (7) depend on the microstructure parameter *l*. Thus this model makes it possible to take into account the effect of the microstructure size on the free vibrations of thin plates.
After neglecting underline terms in (7_2) the following equation for fluctuation amplitudes V^A can be written

$$V^A=-\langle B\partial\partial h^A\rangle\partial\partial W(\langle B\partial\partial h^A\partial\partial h^B\rangle)^{-1}\tag{8}$$

and after the substituting (8) into (7_1) the equation for W is derived

$$\partial\partial\big((\langle B\rangle-\langle B\partial\partial h^A\rangle\langle B\partial\partial h^B\rangle(\langle B\partial\partial h^A\partial\partial h^B\rangle)^{-1})\partial\partial W\big)+\langle\mu\rangle\ddot{W}-\langle\vartheta\rangle\partial\partial\ddot{W}=0.\tag{9}$$

The Equations (8, 9) together with micro-macro decomposition (3) represent the asymptotic model of thin functionally graded plate bands. This model neglects the effect of microstructure size.

3. Example: free vibrations of plate band

3.1. Introduction

Free vibrations of a plate band clamped on both edges with span L along x-axis are considered. The properties of the plate band are

$$\rho(\cdot,z),E(\cdot,z)=\begin{cases}\rho',E' & for\quad z\in\left(\frac{1}{2}(1-\gamma(x))l,\frac{1}{2}(1+\gamma(x))l\right)\\ \rho'',E'' & for\quad z\in\left[0,\frac{1}{2}(1-\gamma(x))l\right]\cup\left[\frac{1}{2}(1+\gamma(x))l,l\right]\end{cases}\tag{10}$$

with a distribution function of material properties $\gamma(x)$.
The considerations are restricted only to one fluctuation shape function, $A = N = 1$. Denote $h\equiv h^1$, $V\equiv V^1$, and the micro-macro decomposition of plate band deflection $w(x, t)$ has the form

$$w(x,t)=W(x,t)+h(x)V(x,t)\tag{11}$$

with the slowly-varying functions $W(\cdot, t)$, $V(\cdot, t)$ for every $t\in(t_0, t_1)$, and the fluctuation shape function $h(\cdot)$.

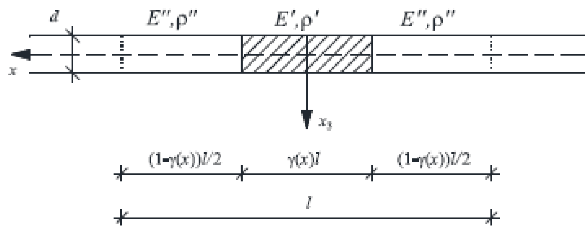

Fig. 2. "Basic cell"

The cell structure of the functionally graded plate band is shown in Fig. 2. The periodic approximation of the fluctuation shape function $h(x)$ has the form

$$\tilde{h}(x,z)=l^2\left[\cos\left(\frac{2\pi z}{l}\right)+c(x)\right]\quad z\in\Delta(z)\quad z\in\Omega.\tag{12}$$

The parameter $c(x)$ is a slowly-varying function in x and determined by $\langle\tilde{\mu}\tilde{h}\rangle=0$

$$c=c(x)=\{sin[\pi\tilde{\gamma}(x)](\rho'-\rho'')\}\{\pi\{\rho'\tilde{\gamma}(x)+\rho''[1-\tilde{\gamma}(x)]\}\}^{-1},\tag{13}$$

where $\tilde{\gamma}(x)$ is the periodic approximation of the distribution function of the material properties $\gamma(x)$. The parameter $c(x)$ is slowly-varying function of x and is treated as constant in the calculations of derivatives $\partial\tilde{h}, \partial\partial\tilde{h}$.

3.2. The Ritz method

Equations (7) and (9) have slowly-varying coefficients and it is difficult to find an analytical solutions of them. Therefore the known Ritz method can be used to derive approximate formulas for free vibrations frequencies. In order to obtain these formulas, it is necessary to determinate the relationship of maximal strain energy $\mathcal{E}_{max}$ and the maximal kinetic energy $\mathcal{K}_{max}$.
Solutions of Equations (7) and Equation (9) are assumed in the form satisfying the boundary conditions for the plate band clamped on both edges

$$
\begin{aligned}
W(x,t) &= A_W\left(U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right)\cos(\omega t) \\
V(x,t) &= A_V\left(U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right)\cos(\omega t)
\end{aligned}
\tag{14}
$$

with the wave number α; the free vibrations frequency ω; the Krylow-Prager functions $U(\alpha x)$ and $V(\alpha x)$, $U(\alpha x)$=0.5[cosh(αx) – cos(αx)], $V(\alpha x)$=0.5[sinh(αx) – sin(αx)]; the amplitude for macrodeflection A_W; the amplitude for the fluctuation amplitude A_V.
Introducing denotations

$$
\begin{aligned}
\breve{B} &= \frac{d^3}{12(1-\nu^2)}\int_0^L\{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
\bar{B} &= \frac{\pi d^3}{3(1-\nu^2)}(E' - E'')\int_0^L \sin\big(\pi\tilde{\gamma}(x)\big)\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
\hat{B} &= \frac{(\pi d)^3}{3(1-\nu^2)}\int_0^L\{(E' - E'')\big[2\pi\tilde{\gamma}(x) + \sin\big(2\pi\tilde{\gamma}(x)\big)\big] + 2\pi E''\}\cdot \\
&\qquad \cdot\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
\breve{\mu} &= d\int_0^L\{[1-\tilde{\gamma}(x)]\rho'' + \tilde{\gamma}(x)\rho'\}\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
\bar{\mu} &= \frac{d}{4\pi}\int_0^L\{(\rho' - \rho'')\big[2\pi\tilde{\gamma}(x) + \sin\big(2\pi\tilde{\gamma}(x)\big)\big] + 2\pi\rho''\}\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
&\quad + \frac{d}{4}(\rho' - \rho'')\int_0^L c(x)\big[\pi c(x)\tilde{\gamma}(x) - 2\sin\big(\pi\tilde{\gamma}(x)\big)\big]\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
&\quad + d\rho''\int_0^L [c(x)]^2\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
\breve{\vartheta} &= \frac{d^3}{12}\int_0^L\{[1-\tilde{\gamma}(x)]\rho'' + \tilde{\gamma}(x)\rho'\}\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx \\
\bar{\vartheta} &= \frac{\pi d^3}{12}\int_0^L\{(\rho' - \rho'')\big[2\pi\tilde{\gamma}(x) - \sin\big(2\pi\tilde{\gamma}(x)\big)\big] + 2\pi\rho''\}\left[U(\alpha x) - \frac{\cosh(\alpha L) - \cos(\alpha L)}{\sinh(\alpha L) - \sin(\alpha L)}V(\alpha x)\right]^2 dx
\end{aligned}
\tag{15}
$$

and using (14), the formulas of the maximum strain energies and kinetic energies of the tolerance model take the form

$$
\mathcal{E}_{max}^{TM} = \tfrac{1}{2}\big(\breve{B}A_W^2\alpha^4 + 2\bar{B}A_W A_V\alpha^2 + \hat{B}A_V^2\big), \qquad \mathcal{K}_{max}^{TM} = \tfrac{1}{2}\big[\big(\breve{\mu} + \breve{\vartheta}\alpha^2\big)A_W^2 + l^2\big(\bar{\mu}l^2 + \bar{\vartheta}\big)A_V^2\big]\omega^2. \tag{16}
$$

The conditions of the Ritz method have the form

$$\frac{\partial(\mathcal{E}_{max}-\mathcal{K}_{max})}{\partial A_w}=0,\quad \frac{\partial(\mathcal{E}_{max}-\mathcal{K}_{max})}{\partial A_V}=0. \tag{17}$$

Using (17) to relations (16), after some manipulations, the following formulas are obtained

$$(\omega_{-,+})^2=\frac{\alpha^4 l^2(\bar{\mu}l^2+\bar{\vartheta})\breve{B}+(\breve{\mu}+\breve{\vartheta}\alpha^2)\hat{B}}{2l^2(\bar{\mu}l^2+\bar{\vartheta})(\breve{\mu}+\breve{\vartheta}\alpha^2)}+ \\ \mp\frac{\sqrt{\left[\alpha^4 l^2(\bar{\mu}l^2+\bar{\vartheta})\breve{B}-(\breve{\mu}+\breve{\vartheta}\alpha^2)\hat{B}\right]^2+4\alpha^4 l^2(\bar{\mu}l^2+\bar{\vartheta})(\breve{\mu}+\breve{\vartheta}\alpha^2)\bar{B}^2}}{2l^2(\bar{\mu}l^2+\bar{\vartheta})(\breve{\mu}+\breve{\vartheta}\alpha^2)}, \tag{18}$$

for the lower ω_- and the higher ω_+ free vibrations frequencies of the tolerance model.
For the asymptotic model, the expressions for maximum energies are written as

$$\mathcal{E}^{AM}_{max}=\tfrac{1}{2}(\breve{B}A_W^2\alpha^4+2\bar{B}A_W A_V\alpha^2+\hat{B}A_V^2),\qquad \mathcal{K}^{TM}_{max}=\tfrac{1}{2}(\breve{\mu}+\breve{\vartheta}\alpha^2)A_W^2\omega^2. \tag{19}$$

Applying conditions (17) to equations (19), after some manipulations, the following formula is obtained

$$\omega^2=\frac{\breve{B}\hat{B}-\bar{B}^2}{(\breve{\mu}+\breve{\vartheta}\alpha^2)\hat{B}}\alpha^4 \tag{20}$$

of the lower ω free vibrations frequency.

3.3. Results

Calculations are made for the following distribution functions of the material properties $\gamma(x)$

$$\tilde{\gamma}(x)=\sin^2\left(\frac{\pi x}{L}\right),\quad \tilde{\gamma}(x)=\cos^2\left(\frac{\pi x}{L}\right),\quad \tilde{\gamma}(x)=\left(\frac{x}{L}\right)^2,\quad \tilde{\gamma}(x)=\sin\left(\frac{\pi x}{L}\right),\quad \tilde{\gamma}(x)=0.5. \tag{21}$$

Let us introduce dimensionless of lower frequency parameters for the tolerance model and the asymptotic model, respectively

$$\Omega_-^2=\frac{12(1-\nu^2)\rho'}{E'}L^2(\omega_-)^2,\quad \Omega^2=\frac{12(1-\nu^2)\rho'}{E'}L^2\omega^2, \tag{22}$$

where ω_-, ω are the free vibration frequencies determined by Equations (18_1) and (20).

Results of calculations for the plate band clamped on both edges are shown in Figs. 3 – 7. Calculations are made for Poisson's ratio ν = 0.3, wave number α = 4.73 (which corresponds to the first mode of free vibrations of the homogeneous plate band clamped on both edges), ratio of plate thickness d/l = 0.1. Figures present plots of lower frequency parameters for different distribution functions of material properties versus ratio of Young's modulus E''/E' or ratio mass density ρ''/ρ'.

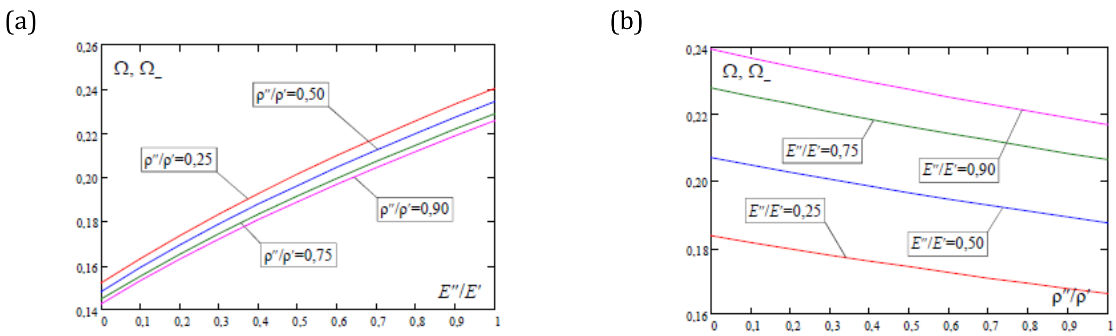

Fig. 3. Plots of the frequency parameters Ω_- and Ω for $\tilde{\gamma}(x)=\sin^2\left(\frac{\pi x}{L}\right)$, depending on parameters: a) E''/E', b) ρ''/ρ'.

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(a) (b)

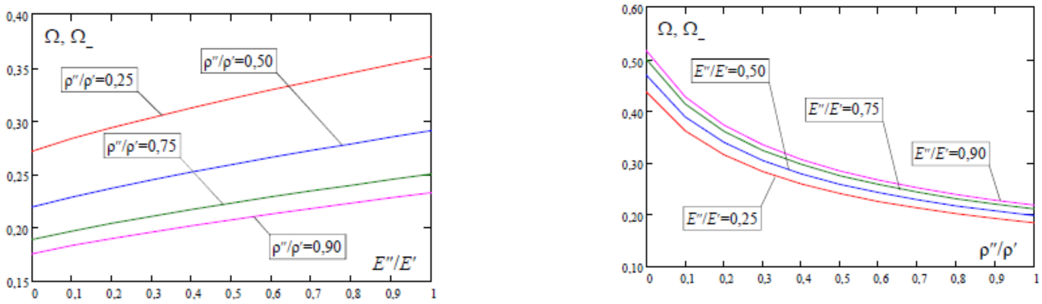

Fig. 4. Plots of frequency parameters $\Omega\_$ and Ω for $\tilde{\gamma}(x) = \cos^2\left(\frac{\pi x}{L}\right)$, depending on parameters: a) E''/E', b) ρ''/ρ'.

(a) (b)

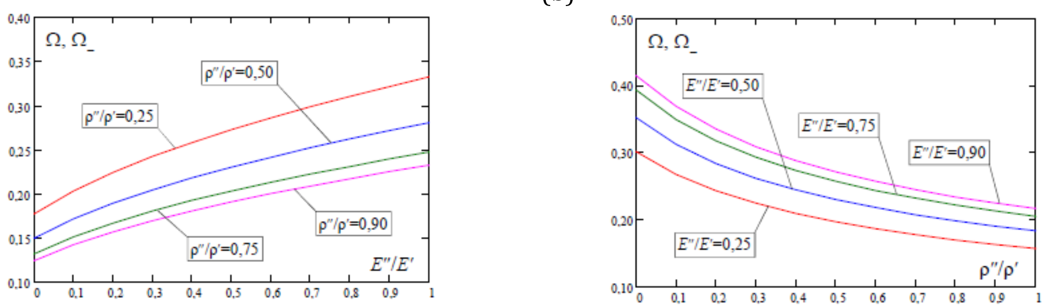

Fig. 5. Plots of the frequency parameters $\Omega\_$ and Ω for $\tilde{\gamma}(x) = \left(\frac{x}{L}\right)^2$, depending on parameters: a) E''/E', b) ρ''/ρ'.

(a) (b)

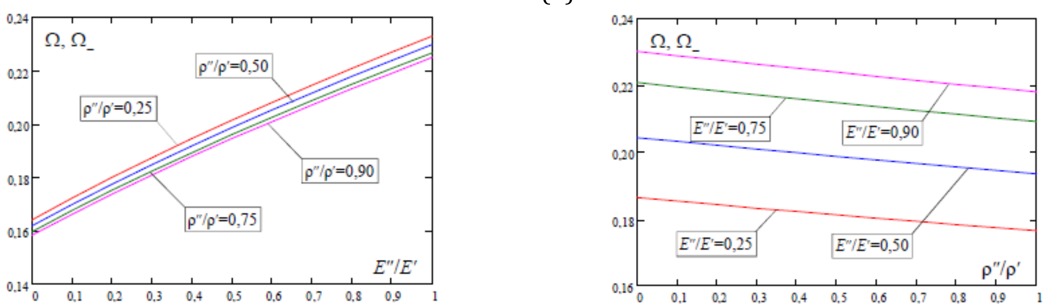

Fig. 6. Plots of the frequency parameters $\Omega\_$ and Ω for $\tilde{\gamma}(x) = \sin\left(\frac{\pi x}{L}\right)$, depending on parameters: a) E''/E', b) ρ''/ρ'.

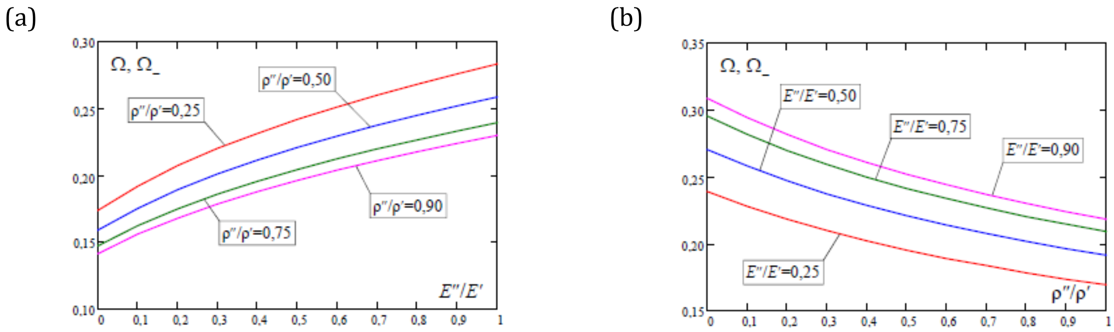

Fig. 7. Plots of the frequency parameters $\Omega\_$ and Ω for $\tilde{\gamma}(x) = 0.5$ depending on parameters: a) E''/E', b) ρ''/ρ'.

4. Conclusions

In this note the main aim is to shown an application of the tolerance and the asymptotic models to analyse free vibrations of thin microstructured functionally graded plate bands clamped of both edges.

The tolerance model is obtained using the tolerance modelling method to the known differential equations of thin microstructured functionally graded plate bands. This method leads from the differential equation with non-continuous, highly oscillating and tolerance-periodic coefficients to the system of differential equations with slowly-varying coefficients. The tolerance model makes it possible to describe the effect of the microstructure size on the overall behaviour of these plates. The asymptotic model is formulated using the asymptotic modelling procedure, shown in [8, 23], or can be also obtained from the tolerance model equations by neglecting suitable terms. However, the asymptotic model describes only the behaviour of these plates on the macrolevel.

In the example for the plate band clamped of both edges free vibrations frequencies have been analysed for various distribution functions of material properties $\gamma(x)$ and different ratios of material properties E''/E' and ρ''/ρ'.

Analysing results of this example it can be observed that:

- lower free vibrations frequencies can be analysed using the both presented models – the tolerance and the asymptotic,
- the frequencies of lower free vibrations decrease with increasing of ratio ρ''/ρ' and they increase with increasing of ratio E''/E',
- using the different distribution functions of the material properties $\gamma(x)$ it can be made microstructure plates having lower fundamental free vibrations frequencies which are smaller or higher than these frequencies for the homogenous plate made of the stronger material for different pairs of ratios (E''/E', ρ''/ρ').

Hence, the tolerance model can be used as a tool to analyse various problems of vibration of thin functionally graded plates with microstructure, which are heterogeneous in planes parallel to the midplane. It should be emphasized that the tolerance model allows to analyse not only lower order fundamental vibrations corresponding to the macrostructure, but also higher order vibrations corresponding to the microstructure.

Additional information

The authors declare no competing financial interests.

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